1. [FIS P58 33]. This problem does not assume that W_1 and W_2 are finite-dimensional. So you should never assume $\beta_1 = \{v_1, \cdots, v_n\}$ (same for β_2). All you have are

$$span(\beta_{1,2}) = W_{1,2}$$

and $\beta_{1,2}$ are linear independent sets. Also a direct sum $W_1 \bigoplus W_2$ means $W_1 \cap W_2 = \{0\}$ but not $W_1 \cap W_2 = \emptyset$. Indeed since W_1 and W_2 are both subspaces, they both contain 0 vector and their intersection contains at least the zero vector. We now turn to the proof.

Proof. (a) Because the sum $W_1 \bigoplus W_2$ is a direct sum, by definition, $W_1 \cap W_2 = \{0\}$. Because $span(\beta_{1,2}) = W_{1,2}$, in particular, $\beta_{1,2} \subset W_{1,2}$ and then we have

$$\beta_1 \cap \beta_2 \subset W_1 \cap W_2 = \{0\}.$$

As bases of vector spaces, $\beta_{1,2}$ do not contain zero vector. So $\beta_1 \cap \beta_2 = \emptyset$. To prove $\beta_1 \cup \beta_2$ is a basis for V, we need to show (1) $V \subset span(\beta_1 \cup \beta_2)$; (2) $\beta_1 \cup \beta_2$ is a linear independent set.
(1): For a $v \in V = W_1 \bigoplus W_2$, we can find $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$. Because $span(\beta_{1,2}) = W_{1,2}$, we can find $\{w_1^1, \cdots, w_n^1\} \subset \beta_1$ and $\{w_1^2, \cdots, w_m^2\} \subset \beta_2$ such that

$$w_1 = \sum_{i=1}^{n} a_i w_i^1, \quad w_2 = \sum_{j=1}^{m} b_j w_j^2.$$

Therefore, $v = \sum_{i=1}^{n} a_i w_i^1 + \sum_{j=1}^{m} b_j w_j^2 \in span(\beta_1 \cup \beta_2)$.
(2): To show $\beta_1 \cup \beta_2$ is a linear independent set, we need to show any finite selection of distinct vectors from $\beta_1 \cup \beta_2$ are linear independent. If the finite selection of vectors are all from β_1 or β_2, by linear independency of β_1 and β_2, we are done. We then assume that the finite selection of vectors contain both vectors from β_1 and β_2 say $\{w_1^1, \cdots, w_n^1\} \subset \beta_1$ and $\{w_1^2, \cdots, w_m^2\} \subset \beta_2$. And, we need to show

$$\sum_{i=1}^{n} a_i w_i^1 + \sum_{j=1}^{m} b_j w_j^2 = 0$$

has only trivial solution. Rearranging terms yields

$$\sum_{i=1}^{n} a_i w_i^1 = -\sum_{j=1}^{m} b_j w_j^2 =: v$$

Because $v = \sum_{i=1}^{n} a_i w_i^1 \in W_1$ and $v = -\sum_{j=1}^{m} b_j w_j^2 \in W_2$, we have $v \in W_1 \cap W_2 = \{0\}$. Therefore,

$$\sum_{i=1}^{n} a_i w_i^1 = -\sum_{j=1}^{m} b_j w_j^2 = v = 0,$$

which implies

$$a_1 = a_2 = \cdots = a_n = 0, \quad b_1 = b_2 = \cdots = b_m = 0$$

by linear independency of β_1 and β_2.
(b) Because $\beta_1 \cup \beta_2$ is a basis for V, $\forall v \in V$, $\exists \{w_1^1, \cdots, w_n^1\} \subset \beta_1$ and $\{w_1^2, \cdots, w_m^2\} \subset \beta_2$, s.t. $v = \sum_{i=1}^n a_i w_i^1 + \sum_{j=1}^m b_j w_j^2$. But $\sum_{i=1}^n a_i w_i^1 \in W_1$ and $\sum_{j=1}^m b_j w_j^2 \in W_2$, so $V \subset W_1 + W_2$. Now let $v \in W_1 \cap W_2$, $v \in W_1 \Rightarrow v = \sum_{i=1}^n a_i w_i^1$ for some $\{w_1^1, \cdots, w_n^1\} \subset \beta_1$ and $v \in W_2 \Rightarrow v = \sum_{j=1}^m b_j w_j^2$ for some $\{w_1^2, \cdots, w_m^1\} \subset \beta_2$. Now

$$0 = v - v = \sum_{i=1}^n a_i w_i^1 - \sum_{j=1}^m b_j w_j^2$$

and by linear independency of $\beta_1 \cup \beta_2$

$$a_1 = a_2 = \cdots = a_n = 0, \quad b_1 = b_2 = \cdots = b_m = 0.$$

Hence, $v = 0$ and $V = W_1 \bigoplus W_2$. □

2. [FIS P58 34]. (a) Expand a basis β_1 of W_1 to a basis $\beta_1 \cup \beta_2$ of V and define $W_2 = span(\beta_2)$. Then apply 33 part (b).

   (b) $W_2 = span(\{(0,1)\})$ and $W_2' = span(\{(1,1)\})$