Please complete following problems.

1. Excise 6.1 (a) (10 points)
2. Excise 6.1 (g) (10 points)
3. Excise 6.5. (For part c, omit "What happens to the solution near $T = T_c$ is ...") (20 points)

3. (a). A steady state y_s must solve $\mu y_s = e^{-y_s}$. Clearly for any $\mu > 0$ there is a unique solution $y_s = y_s(\mu)$ corresponding to the intersection point of linear function μy_s and exponential function e^{-y_s}. As μ increases from 0 to ∞, $y_s(\mu)$ decreases strictly from ∞ to 0. To determine the stability, we find the linearized equation reads

$$y' = -(\mu + e^{-y_s})y = -\mu(1 + y_s)y$$

where $-\mu(1 + y_s) < 0$, whence the steady state y_s is asymptotic stable.

(b). Since $\mu = y_s^{-1}e^{-y_s}$, from now on we parameterize the equation by y_s. For the delayed equation with $T > 0$, we linearize the equation to obtain

$$y' = -y_s^{-1}e^{-y_s}y_1 - e^{-y_s}y(t - T).$$

Substituting $y = e^{\gamma t}$ yields the characteristic equation

$$\gamma = -y_s^{-1}e^{-y_s} - e^{-y_s}e^{-\gamma T}.$$

With $\gamma := \gamma_r + i\gamma_i$, $\gamma_r, \gamma_i \in \mathbb{R}$, we find the characteristic equation amounts to $\mathbf{F}(\gamma_r, \gamma_i) = \mathbf{0}$ where

$$\mathbf{F}(\gamma_r, \gamma_i; y_s, T) := \begin{pmatrix} \gamma_r + e^{-y_s}\left(y_s^{-1} + e^{-\gamma_r T}\cos(\gamma_i T)\right) \\ \gamma_i - e^{-y_s}e^{-\gamma_r T}\sin(\gamma_i T) \end{pmatrix}.$$

Clear is that $\mu > 1/e$ amounts to $y_s < 1$. If $y_s < 1$ we claim that any solution (γ_r, γ_i) of $\mathbf{F}(\gamma_r, \gamma_i) = \mathbf{0}$ must satisfy $\gamma_r < 0$. For if $\gamma_r \geq 0$, we have

$$\mathbf{F}_1(\gamma_r, \gamma_i; y_s, T) = \gamma_r + e^{-y_s}\left(y_s^{-1} + e^{-\gamma_r T}\cos(\gamma_i T)\right) \geq e^{-y_s}\left(y_s^{-1} + e^{-\gamma_r T}\cos(\gamma_i T)\right) > 0.$$

Contradiction.

(c). Let us determine stability boundaries in the parameter space $\{(y_s, T) : y_s, T > 0\}$ where $\mathbf{F}(\gamma_r, \gamma_i; y_s, T)$ achieves neutral roots with $\gamma_r = 0$. This readily yields

$$e^{-y_s}\left(y_s^{-1} + \cos(\gamma_i T)\right) = 0, \quad \gamma_i - e^{-y_s}\sin(\gamma_i T) = 0.$$

For $y_s > 1$, the foregoing equation admits solutions satisfying $\cos(\gamma_i T) = -y_s^{-1}$. Inserting it into the latter equation we find

$$\gamma_{i,\pm} = \pm e^{-y_s}\sqrt{1 - y_s^{-2}}$$

and the stability boundaries satisfy

$$-\sqrt{y_s^2 - 1} = \tan\left(e^{-y_s}\sqrt{1 - y_s^{-2}}\,T\right).$$

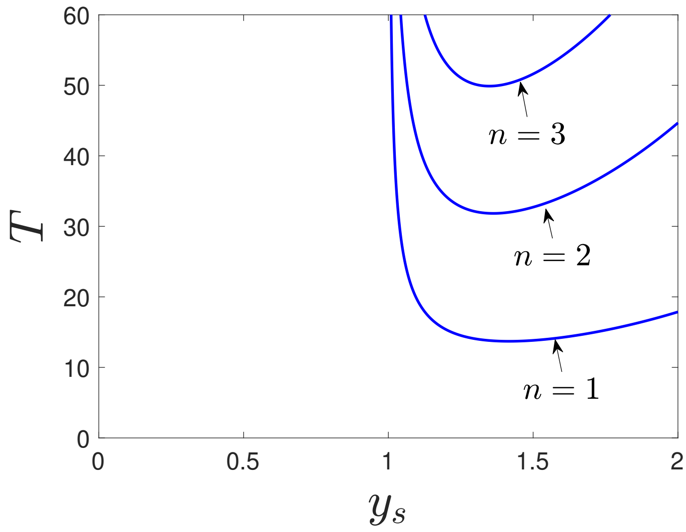

Figure 1: Stability boundaries.

The latter equations gives a families of boundaries

$$T_n(y_s) = \frac{n\pi - \arctan\left(\sqrt{y_s^2 - 1}\right)}{e^{-y_s}\sqrt{1 - y_s^{-2}}} = \frac{n\pi - \arctan\left(\sqrt{y_s^2 - 1}\right)}{\mu\sqrt{y_s^2 - 1}}, \quad n \geq 1, y_s > 1.$$

For $y_s \in (1, \infty)$, there holds $T_1(y_s) < T_2(y_s) < \cdots < T_n(y_s) < \cdots$. Also $T_n(1^+), T_n(+\infty) = \infty$. We plot the boundaries in Figure 1. Since the region

$$\{(y_s, T) : 1 < y_s, 0 < T < T_1(t_s)\}$$

is connected with the region $\{(y_s, T) : y_s < 1\}$ where asymptotic stability holds, we immediately obtain asymptotic stability of the aforementioned region. Next, we determine the stability in the region $\{(y_s, T) : 1 < y_s, T_1(t_s) < T\}$. We compute by implicit function theorem that

$$\begin{aligned}\begin{pmatrix} \frac{\partial \gamma_r}{\partial T} \\ \frac{\partial \gamma_i}{\partial T} \end{pmatrix}(y_s, T_n(y_s)) &= -\left[\begin{pmatrix} \frac{\partial \mathbf{F}_1}{\partial \gamma_r} & \frac{\partial \mathbf{F}_1}{\partial \gamma_i} \\ \frac{\partial \mathbf{F}_2}{\partial \gamma_r} & \frac{\partial \mathbf{F}_2}{\partial \gamma_i} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \mathbf{F}_1}{\partial T} \\ \frac{\partial \mathbf{F}_2}{\partial T} \end{pmatrix}\right](0, \gamma_{i,\pm}, y_s, T_n(y_s)) \\ &= \begin{pmatrix} e^{-2y_s}(1 - y_s^{-2})\left(1 + 2T_n(y_s) y_s^{-1} e^{-y_s}\right) \\ * \end{pmatrix}.\end{aligned}$$

where we find

$$\frac{\partial \gamma_r}{\partial T}(y_s, T_n(y_s)) > 0, \quad \text{for any } n \geq 1 \text{ and } y_s > 1.$$

Thus when (y_s, T) crosses the stability boundaries, there is always a pair of complex conjugate eigenvalues with non-zero imaginary parts crossing the imaginary axis with non-zero speeds from the stable side to the unstable side as T increases. This gives a transversal Hopf bifurcation and also yields instability in the region $\{(y_s, T) : 1 < y_s, T_1(t_s) < T\}$.