Please complete following problems.

1. Recall from Lecture 7 page 6.

   *We rigorously justify the validity of the solution of the form*

   $$b_0 q(x)^{-1/4} e^{\frac{1}{\varepsilon}\int^x \sqrt{q(s)}ds} (1+O(\varepsilon)).$$

   *The other part can be obtained by working with $\tilde{\Phi} := w_2/w_1$. We also note that in this case we shall integrate in the backward direction when defining the contraction mapping to gain smallness from the exponential part in the case $q > 0$. Such solution is independent from the foregoing one because at $x = 0$ $|w_2(0)/w_1(0)| < \infty$ while the ratio of the foregoing one is infinite.*

   Define $\Psi(s) := w_2(s)/w_1(s)$ and compute the ODE satisfied by $\Psi(s)$. Impose the initial condition $\Psi(L/\varepsilon) = 0$ and write down the corresponding integral equation which leads to the definition of an operator on

   $$X := \{\Psi : \Psi \in C^b([0, L/\varepsilon], \mathbb{C}),\ \Psi(L/\varepsilon) = 0\}.$$

   (5 points).

   Rigorously justify the validity of the other solution of the form

   $$a_0 q(x)^{-1/4} e^{\frac{1}{\varepsilon}\int^x -\sqrt{q(s)}ds}\big(1+O(\varepsilon)\big).$$

   by working out the sections 4.1, 4.2, and 4.3 from Lecture 7 for $\Psi$. (5 points for each section.)

2. Excise 4.1 (b) (10 points)
3. Excise 4.16 (c) (10 points)

2. Note that the imhomogeneous equation

$$\varepsilon^2 y'' + \varepsilon x y' - y = -1$$

admits a particular solution $y_p = 1$. Setting $y_h = y - y_p$, we obtain that $y_h$ solve the BVP

$$\begin{cases} \varepsilon^2 y'' + \varepsilon x y' - y = 0 \\ y(0) = -1 \quad y(1) = 2. \end{cases}$$

Consider the WKB ansatz

$$y_h(x) \sim e^{\theta(x)/\varepsilon^\alpha}[y_0(x) + \varepsilon y_1(x) + \cdots]$$

with

$$\begin{aligned} y_h' &\sim e^{\theta(x)/\varepsilon^\alpha}[\varepsilon^{-\alpha}\theta' y_0 + y_0' + \theta' y_1 + \cdots] \\ y_h'' &\sim e^{\theta(x)/\varepsilon^\alpha}[\varepsilon^{-2\alpha}(\theta')^2 y_0 + \varepsilon^{-\alpha}(\theta'' y_0 + 2\theta' y_0' + (\theta')^2 y_1) + \cdots]. \end{aligned}$$

Inserting the ansatz into the differential equation yields that

$$e^{\theta(x)/\varepsilon^\alpha}[\varepsilon^{-2\alpha+2}(\theta')^2 y_0 + \varepsilon^{-\alpha+1} x\theta' y_0 - y_0 + \cdots] = 0.$$

To balance the lowest order terms, it requires that $\alpha = 1$ and

$$(\theta')^2 + x\theta' - 1 = 0$$

whence

$$\theta'_\pm = \frac{-x \pm \sqrt{x^2+4}}{2}.$$

With $\theta_\pm(0) = 0$, we find that

$$\theta_\pm(x) = \int_0^x \frac{-s \pm \sqrt{s^2+4}}{2} ds = \pm\left(\mathrm{a}sinh(\frac{x}{2}) + \frac{x\sqrt{x^2+4}}{4}\right) - \frac{x^2}{4}.$$

At $\mathcal{O}(|\varepsilon|)$ order, we obtain that

$$0 = \theta''_\pm y_0 + 2\theta'_\pm y'_0 + (\theta'_\pm)^2 y_1 + x(y'_0 + \theta'_\pm y_1) - y_1 = \theta''_\pm y_0 + 2\theta'_\pm y'_0 + x y'_0$$

so that

$$y'_0 = -\frac{\theta''_\pm}{2\theta'_\pm + x} y_0 = \frac{-x \pm \sqrt{x^2+4}}{2(x^2+4)} y_0.$$

Solving the equation we get

$$y_{0,+}(x) = e^{asinh(x/2)/2 + \ln(2)/2 - \ln(x^2+4)/4} a, \quad y_{0,-}(x) = e^{-asinh(x/2)/2 + \ln(2)/2 - \ln(x^2+4)/4} b.$$

The general solution to the homogeneous equation reads

$$\begin{aligned} y_h(x) = &e^{\theta_+(x)/\varepsilon} e^{asinh(x/2)/2 + \ln(2)/2 - \ln(x^2+4)/4} a \\ &+ e^{\theta_-(x)/\varepsilon} e^{-asinh(x/2)/2 + \ln(2)/2 - \ln(x^2+4)/4} b + \cdots \end{aligned}$$

**Boundary conditions.** At $x = 0$, we require that

$$a + b = -1.$$

At $x = 1$, we compute

$$\theta_+(1) = asinh(\frac{1}{2}) + \frac{\sqrt{5}-1}{4} > 0, \quad \theta_-(1) = -asinh(\frac{1}{2}) - \frac{\sqrt{5}+1}{4} < 0$$

So we require that

$$e^{\theta_+(1)/\varepsilon} e^{asinh(1/2)/2 + \ln(2)/2 - \ln(5)/4} a \approx 2.$$

3. Rewrite the equation as

$$\varepsilon^2 y'' - q(x) y = 0,$$

where

$$q(x) = -x(x+3)^2.$$

The turning points are the zeros of $q(x)$:

$$q(x) = 0 \Longrightarrow x = 0 \text{ or } x = -3.$$

Only $x = 0$ lies in the interval $[-1,1]$. This is a **simple turning point** because

$$q'(0) = -9 < 0.$$

By equations (4.57), (4.49) , and (4.50) of Holmes's, we find

$$y(x) \sim \frac{1}{q(x)^{1/4}}\left[A_L e^{\frac{1}{\varepsilon}\int_x^0 \sqrt{q(s)}\,ds} + B_L e^{-\frac{1}{\varepsilon}\int_x^0 \sqrt{q(s)}\,ds}\right], \quad x<0,$$
$$y(x) \sim \frac{1}{|q(x)|^{1/4}}\left[2B_L\cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{|q(s)|}\,ds - \frac{\pi}{4}\right) + A_L\cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{|q(s)|}\,ds + \frac{\pi}{4}\right)\right], \quad x>0.$$

**Boundary conditions.** At $x = -1$, we compute

$$q(-1)^{1/4} = \sqrt{2}, \quad \int_{-1}^0 \sqrt{q(s)}\,ds = \int_{-1}^0 \sqrt{-s}\,(s+3)\,ds = \frac{8}{5}.$$

The boundary condition $y(-1) = 0$ gives

$$\frac{1}{\sqrt{2}}\left[A_L e^{\frac{8}{5\varepsilon}} + B_L e^{-\frac{8}{5\varepsilon}}\right] = 0 \Longrightarrow A_L = -B_L e^{-16/(5\varepsilon)}. \tag{1}$$

At $x = 1$, we compute

$$|q(1)|^{1/4} = 2 \quad \theta_1 = \int_0^1 \sqrt{|q(s)|}\,ds = \int_0^1 \sqrt{s}\,(s+3)\,ds = \frac{12}{5}.$$

The boundary condition $y(1) = 1$ gives

$$\frac{1}{2}\left[2B_L\cos\left(\frac{12}{5\varepsilon} - \frac{\pi}{4}\right) + A_L\cos\left(\frac{12}{5\varepsilon} + \frac{\pi}{4}\right)\right] = 1. \tag{2}$$

Inserting (1) into (2) yields

$$B_L\left[2\cos\left(\frac{12}{5\varepsilon} - \frac{\pi}{4}\right) - e^{-16/(5\varepsilon)}\cos\left(\frac{12}{5\varepsilon} + \frac{\pi}{4}\right)\right] = 2.$$

For $\varepsilon \ll 1$, the exponential term is negligible, so

$$B_L \approx \sec\left(\frac{12}{5\varepsilon} - \frac{\pi}{4}\right), \quad A_L \approx -B_L e^{-16/(5\varepsilon)}.$$