Please complete following problems.

1. Design an algorithm computing $x_n(\varepsilon)$ (see page 3 of the lecture notes) inductively. (There are multiple ways. Design and describe one. ) (10 points)
2. Use your favorite method to give a proof of Proposition 2 on page 3. (10 points)
3. In the proof of Theorem 1, convince yourself that the fixed point of the map $\mathscr{T}_\varepsilon$ is $C^2$ and indeed solves the origin initial value problem. (integral solution $\Rightarrow$ strong solution) (0 points, no need for hand-writing solutions)
4. Define a sequence of functions by $\{y_n(t)\}_{n=1}^{\infty}$ by $y_0(t) = x_0(t)$ and $y_n(t;\varepsilon) = (\mathscr{T}_\varepsilon y_{n-1}(\cdot;\varepsilon))(t)$, $n \geq 1$.
   i. Show that $\|x_\varepsilon(\cdot) - y_n(\cdot;\varepsilon)\| = \mathscr{O}(|\varepsilon|^{n+1})$ where $x_\varepsilon(\cdot)$ is the fixed point of $\mathscr{T}_\varepsilon$. (5 points)
   ii. Compute the $\mathscr{O}(1)$- and $\mathscr{O}(|\varepsilon|)$- terms of $y_1(\cdot;\varepsilon)$ and call them $x_0(t)$ and $x_1(t)\varepsilon$. Check they agree the results obtained from formal computations. (5 points)
   iii. Show that $\|y_1(\cdot;\varepsilon) - x_0(\cdot) - x_1(\cdot)\varepsilon\| = \mathscr{O}(|\varepsilon|^2)$. Combining with i. (with $n = 1$), we finally obtain that
   $$\|x_\varepsilon(\cdot) - x_0(\cdot) - x_1(\cdot)\varepsilon\| = \mathscr{O}(|\varepsilon|^2),$$
   completing the proof of Theorem 1. (10 points)