Introduction to Bifurcation and Stability

1 Motivation: Multiple Solutions in Nonlinear Problems

In nonlinear problems, we usually encountered with multiple solutions. We will study:

- When do multiple solutions appear? (**Bifurcation**)
- Which solutions are physically achievable? (**Stability**)

2 Model Example: Nonlinear Oscillator

Consider the Duffing-type oscillator:

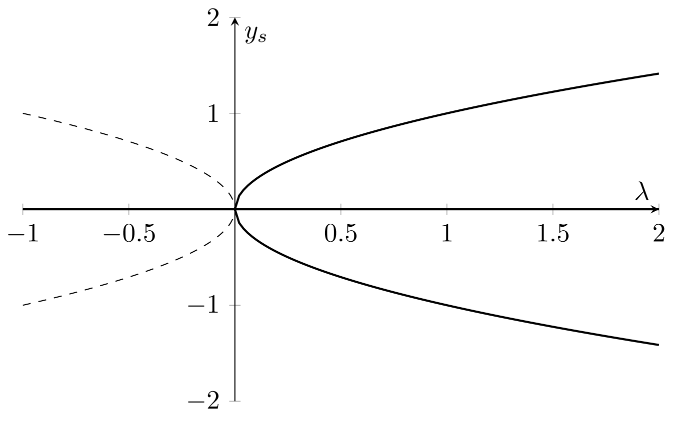
$$y'' + 2\beta y' - \lambda y + y^3 = 0. \tag{6.1}$$

Steady State Analysis. Set derivatives to zero:

$$\lambda y - y^3 = 0 \quad \Rightarrow \quad y(\lambda - y^2) = 0. \tag{6.2}$$

Solutions:

- $y_s = 0$ for all λ
- $y_s = \pm\sqrt{\lambda}$ for $\lambda \geq 0$



This is a **pitchfork bifurcation** at $(\lambda_b, y_b) = (0, 0)$.

2.1 Numerical Observations

Numerical experiments show:

- For $\lambda = 1$: Solutions approach $y_s = \pm 1$, not $y_s = 0$;
- For $\lambda < 0$: Only $y_s = 0$ can be achieved;
- Stability changes at $\lambda = 0$.

3 Bifurcation Analysis

3.1 Problem Setup

Given:

$$F(\lambda, y) = 0 \tag{6.4}$$

where F is smooth in λ and y.

Assume we know one solution branch $y_s(\lambda)$. We seek intersection points with other branches.

Theorem 3.1 (Implicit Function Theorem). *Given $F \in C^1$ with $F(\lambda_0, y_0) = 0$ and $\partial_y F(\lambda_0, y_0) \neq 0$, then for λ near λ_0:*

1. *There exists a unique solution $y = y_s(\lambda)$ with $y_s(\lambda_0) = y_0$*
2. *If $F \in C^k$, then $y_s \in C^k$*

At a bifurcation point (λ_b, y_b):

$$F(\lambda_b, y_b) = 0 \quad \text{and} \quad \partial_y F(\lambda_b, y_b) = 0. \tag{6.5, 6.6}$$

However, these conditions are **necessary but not sufficient** for bifurcation.

Counterexample. $F(\lambda, y) = y^2$.

- **Known solution**: $y_s = 0$ for all λ
- **Conditions check**:

$$F(\lambda, 0) = 0, \qquad \partial_y F(\lambda, 0) = 2y|_{y=0} = 0$$

- **But**: Only one solution exists - no bifurcation occurs!

3.2 Lyapunov-Schmidt Method for Local Bifurcation

Given $y_s(\lambda)$, find bifurcation points and intersecting solutions near $\lambda = \lambda_b$.

Assume the special form:

$$F(\lambda, y) = \lambda y + f(y). \tag{6.7}$$

with $f(0) = f'(0) = 0$, so that for arbitrary λ, $y_s = 0$ is a solution. We compute $\partial_y F(\lambda, y) = \lambda + f'(y)$, yielding that $(\lambda_b, y_b) = (0, 0)$ could be a bifurcation point.

Expansion. Let $\varepsilon = \lambda$ and assume:

$$y \sim \varepsilon^{\alpha} y_1 \tag{6.10}$$

with $\alpha > 0$, $y_1 \neq 0$.

Let n be the smallest positive integer such that $f^{(n)}(0) \neq 0$. Taylor expansion gives:

$$F(\lambda, y) = \lambda y + \frac{1}{n!} y^n f^{(n)}(0) + \cdots .$$

For nonzero solutions:

$$\lambda + \frac{1}{n!} y^{n-1} f^{(n)}(0) + \cdots = 0. \tag{6.8}$$

Substitute expansions:

$$\varepsilon + \frac{1}{n!} y_1^{n-1} \varepsilon^{(n-1)\alpha} f^{(n)}(0) + \cdots = 0. \tag{6.11}$$

Balancing terms, we must set $(n-1)\alpha = 1 \Rightarrow \alpha = \frac{1}{n-1}$, and the equation becomes

$$y_1^{n-1} = -\frac{n!}{f^{(n)}(0)} \Rightarrow y_1 = \left(-\frac{n!}{f^{(n)}(0)}\right)^{\frac{1}{n-1}} .$$

Putting back, we obtain that

$$y \sim \left(-\frac{n!}{f^{(n)}(0)} \varepsilon\right)^{\frac{1}{n-1}} .$$

Noting that for n being odd, we require that

$$-\frac{n!}{f^{(n)}(0)}\varepsilon \geq 0$$

and for n being even, there is no requirement for the sign of ε.

Explicit formulas.

- For n even:

$$y \sim \left(-\frac{n!}{f^{(n)}(0)}\lambda\right)^{\frac{1}{n-1}}. \tag{6.13}$$

- For n odd:

$$y \sim \begin{cases} \pm\left(-\frac{n!}{f^{(n)}(0)}\lambda\right)^{\frac{1}{n-1}} & \text{for } \lambda \geq 0 \text{ if } f^{(n)}(0) < 0, \\ \pm\left(-\frac{n!}{f^{(n)}(0)}\lambda\right)^{\frac{1}{n-1}} & \text{for } \lambda \leq 0 \text{ if } f^{(n)}(0) > 0. \end{cases} \tag{6.14}$$

Bifurcation Types.

n	$f^{(n)}(0) < 0$	$f^{(n)}(0) > 0$
Odd	Supercritical Pitchfork	Subcritical Pitchfork
Even	Transcritical	Transcritical

4 Linearized Stability Analysis

4.1 General Approach

For the damped oscillator:

$$y'' + 2\beta y' + F(\lambda, y) = 0 \tag{6.15}$$

with $\beta > 0$.

Given steady state y_s, perturb:

$$y(0) = y_s + \alpha_0\delta, \quad y'(0) = \beta_0\delta \tag{6.16}$$

with $\delta \ll 1$.

Assume expansion:

$$y(t) \sim y_s + \delta y_1(t) + \cdots \tag{6.17}$$

4.2 Linearized Equation

Substitute and linearize:

$$\delta y_1'' + 2\beta\delta y_1' + F(\lambda, y_s) + \delta y_1 F_y(\lambda, y_s) + \cdots = 0 \tag{6.18}$$

Since $F(\lambda, y_s) = 0$, the $O(\delta)$ problem is:

$$y_1'' + 2\beta y_1' + F_y(\lambda, y_s)y_1 = 0 \tag{6.19}$$

with $y_1(0) = \alpha_0$, $y_1'(0) = \beta_0$.

4.3 Stability Classification

Solution form:

$$y_1(t) = a_0 e^{r_+ t} + a_1 e^{r_- t} \tag{6.21}$$

where

$$r_\pm = -\beta \pm \sqrt{\beta^2 - F_y(\lambda, y_s)} \tag{6.22}$$

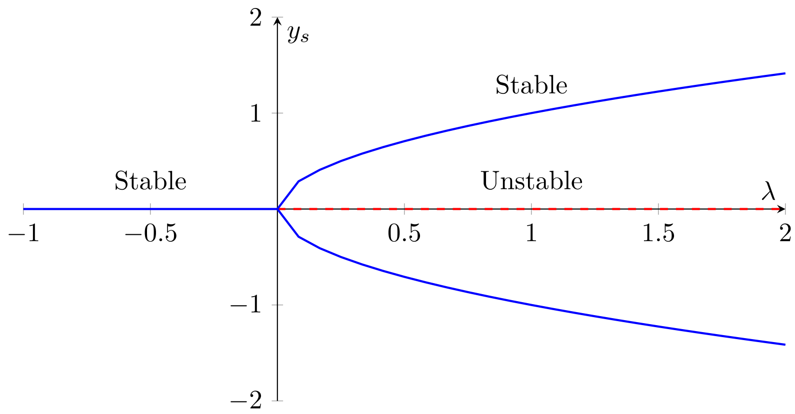
Stability criteria:

- If $F_y(\lambda, y_s) > 0$: $\mathrm{Re}(r_\pm) < 0 \to$ **Asymptotically stable**
- If $F_y(\lambda, y_s) < 0$: $\mathrm{Re}(r_+) > 0 \to$ **Unstable**

4.4 Example: Pitchfork Bifurcation Stability

For $F(\lambda, y) = -\lambda y + y^3$:

- $y_s = 0$: $F_y(\lambda, 0) = -\lambda$
- $y_s = \pm\sqrt{\lambda}$: $F_y(\lambda, \pm\sqrt{\lambda}) = 2\lambda$



Exchange of stability occurs at the bifurcation point.

5 Application: Delay Equation in Population Dynamics

5.1 Problem Setup

Improved population model with gestation period T:

$$y'(t) = \alpha y(t-T) - \beta y^3(t-T). \tag{6.23}$$

Steady states: $y_s = 0$ and $y_s = \sqrt{\alpha/\beta}$ (same as no-delay case, assumed $y_s > 0$).

5.2 Linearized Stability Analysis

Perturb $y_s = \sqrt{\alpha/\beta}$

$$y(t) = y_s + \delta y_1(t) + \cdots .$$

Substituting the ansatz into the delay equation, we obtain

$$\begin{aligned} \delta y_1'(t) &= \alpha(y_s + \delta y_1(t-T) + \cdots) - \beta\left(y_s + \delta y_1(t-T) + \cdots\right)^3 \\ &= \alpha y_s - \beta y_s^3 + \delta\left(\alpha y_1(t-T) - 3\beta y_s^2 y_1(t-T) + \cdots\right) \\ &= -2\delta\alpha y_1(t-T) + \cdots , \end{aligned}$$

yielding

$$y_1'(t) = -2\alpha y_1(t-T). \tag{6.26}$$

Taking $y_1 = e^{rt}$, we obtain the characteristic equation

$$r = -2\alpha e^{-rT} \tag{6.27}$$

Set $r = \gamma + i\omega$ with $\gamma, \omega \in \mathbb{R}$. We find (6.27) amount to $\mathcal{F}(\gamma, \omega; \alpha, T) = \mathbf{0}$ where

$$\mathcal{F}(\gamma, \omega; \alpha, T) := \begin{pmatrix} \gamma + 2\alpha e^{-\gamma T}\cos(\omega T) \\ \omega - 2\alpha e^{-\gamma T}\sin(\omega T) \end{pmatrix}. \tag{6.28}$$

Assume (6.27) or (6.28) achieve roots r_n for $n = 1, 2, \cdots$. The general solution of the linear delay equation (6.26) reads

$$y_1 = \sum_n a_n e^{r_n t},$$

where $r_n = \gamma_n + i\omega_n$ are the solutions to (6.27) or (6.28). The general solution decays to zeros if and only if γ_n for all n.

Stability Boundary. Let $T > 0$ be fixed and α be varied. Set $\gamma = 0$ in (6.28) to find stability transition:
case 1: $\alpha_* = 0$. This implies $\omega_* = 0$.

case 2: $\alpha \neq 0$. We must have

$$\omega_n T = (n - \frac{1}{2})\pi \quad n \in \mathbb{Z}, \quad \text{and, hence} \quad \omega_n = 2\alpha_n(-1)^{n+1}$$

or

$$\alpha_n = (\frac{n}{2} - \frac{1}{4})\frac{\pi}{T}(-1)^{n+1}.$$

Question: when α crosses α_n or α_*, how does the sign of the critical γ change?

Answer: By implicit function theorem, we have

$$\begin{pmatrix} \frac{\partial\gamma}{\partial_\alpha} \\ \frac{\partial\omega}{\partial_\alpha} \end{pmatrix}(0, \omega_n, \alpha_n) = -\left[\begin{pmatrix} \frac{\partial\mathcal{F}_1}{\partial\gamma} & \frac{\partial\mathcal{F}_1}{\partial\omega} \\ \frac{\partial\mathcal{F}_2}{\partial\gamma} & \frac{\partial\mathcal{F}_2}{\partial\omega} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial\mathcal{F}_1}{\partial\alpha} \\ \frac{\partial\mathcal{F}_2}{\partial\alpha} \end{pmatrix}\right](0, \omega_n, \alpha_n)$$
$$= \begin{pmatrix} -\frac{4\,\pi\,(-1)^n\,(2\,n-1)}{4\,\pi^2\,n^2-4\,\pi^2\,n+\pi^2+4} \\ -\frac{8\,(-1)^n}{4\,\pi^2\,n^2-4\,\pi^2\,n+\pi^2+4} \end{pmatrix}$$

At $n = 1$ we have

$$\begin{pmatrix} \frac{\partial\gamma}{\partial_\alpha} \\ \frac{\partial\omega}{\partial_\alpha} \end{pmatrix}(0, \omega_1, \alpha_1) = \begin{pmatrix} \frac{4\,\pi}{\pi^2+4} \\ \frac{8}{\pi^2+4} \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} \frac{\partial\gamma}{\partial_\alpha} \\ \frac{\partial\omega}{\partial_\alpha} \end{pmatrix}(0, 0, 0) = -\left[\begin{pmatrix} \frac{\partial\mathcal{F}_1}{\partial\gamma} & \frac{\partial\mathcal{F}_1}{\partial\omega} \\ \frac{\partial\mathcal{F}_2}{\partial\gamma} & \frac{\partial\mathcal{F}_2}{\partial\omega} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial\mathcal{F}_1}{\partial\alpha} \\ \frac{\partial\mathcal{F}_2}{\partial\alpha} \end{pmatrix}\right](0, 0, 0)$$
$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Our computations show that a critical γ becomes unstable when α increase beyond $\alpha_1 = \frac{\pi}{4T}$ or decreases below $\alpha = 0$. Consequently, the number of negative eigenvalues achieves a local minimum within the interval $\alpha \in (0, \frac{\pi}{4T})$.

Now we show that for small $\alpha > 0$ all solutions (γ, ω) of (6.28) satisfiess $\gamma < 0$. If not, assume the system achieve $\gamma > 0$. By the second equation

$$|\omega| = 2\alpha e^{-\gamma T}|\sin(\omega T)| < 2\alpha.$$

This implies that $\mathcal{F}_1 = \gamma + 2\alpha e^{-\gamma T}\cos(\omega T) > 0$. Contradiction. Stable region: $0 < \alpha < \pi/4$.