The WKB Method – Turning Points

1 Problem Setup and Turning Points

We consider the singular perturbation problem

$$\varepsilon^2 y'' - q(x)y = 0 \tag{4.25}$$

where $\varepsilon \ll 1$ and $q(x)$ is smooth. Recalling from Lecture 7, after setting $s = \frac{x}{\varepsilon}$ and $u(s) = y(x)$, we found that

$$u'' - q(\varepsilon s)u = 0,$$

where we noted the coefficient function $q(\varepsilon s)$ is slowly varying. The second order ODE behaves mostly like a constant-coefficient one. Then, we rewrote it as a first order system

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(\varepsilon s) & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix},$$

where we noted the coefficient matrix can be diagonalized as

$$T \begin{pmatrix} 0 & 1 \\ q(\varepsilon s) & 0 \end{pmatrix} T^{-1} = \Lambda \quad \text{with} \quad T = \begin{pmatrix} -\sqrt{q(\varepsilon s)} & 1 \\ \sqrt{q(\varepsilon s)} & 1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} -\sqrt{q(\varepsilon s)} & 0 \\ 0 & \sqrt{q(\varepsilon s)} \end{pmatrix}.$$

given that $q(\varepsilon s) \neq 0$. At s_t where $q(\varepsilon s_t) = 0$, the coefficient matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is non-diagonalizable and takes the form of a Jardan block.

Turning point At a point x_t where $q(x_t) = 0$. We assume a **simple turning point** $q(x_t) = 0$ but $q'(x_t) \neq 0$. We'll analyze the case $q'(x_t) > 0$, so

- $q(x) < 0$ for $x < x_t$ (oscillatory region).
- $q(x) > 0$ for $x > x_t$ (exponential region).

2 Outer WKB Solutions Away from x_t

From previous WKB analysis, the general solutions are

For $x > x_t$ (exponential region)

$$y_R(x, x_t) = \frac{1}{q(x)^{1/4}} \left[a_R \exp\left(-\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds \right) + b_R \exp\left(\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds \right) \right]. \tag{4.27}$$

For $x < x_t$ (oscillatory region)

$$y_L(x, x_t) = \frac{1}{|q(x)|^{1/4}} \left[a_L \exp\left(-\frac{i}{\varepsilon} \int_{x}^{x_t} \sqrt{|q(s)|} ds \right) + b_L \exp\left(\frac{i}{\varepsilon} \int_{x}^{x_t} \sqrt{|q(s)|} ds \right) \right]. \tag{4.28}$$

We have **4 unknown constants** but only **2 degrees of freedom** in the true solution. The connection comes from the turning point analysis.

3 Transition Layer Analysis at $x = x_t$

3.1 Scaling Determination

Let's determine the appropriate scaling near the turning point. Introduce

$$\bar{x} = \frac{x - x_t}{\varepsilon^\beta} \quad \text{or} \quad x = x_t + \varepsilon^\beta \bar{x} \tag{4.29}$$

where $\beta > 0$ is to be determined.
Expand $q(x)$ near x_t

$$q(x_t + \varepsilon^\beta \bar{x}) = q(x_t) + \varepsilon^\beta \bar{x} q'(x_t) + \mathcal{O}(\varepsilon^{2\beta}) = \varepsilon^\beta \bar{x} q'(x_t) + \mathcal{O}(\varepsilon^{2\beta}) \tag{4.30}$$

since $q(x_t) = 0$.
Let $Y(\bar{x}) = y(x)$ be the solution in the transition layer. Then

$$\frac{dy}{dx} = \varepsilon^{-\beta} \frac{dY}{d\bar{x}}, \quad \frac{d^2y}{dx^2} = \varepsilon^{-2\beta} \frac{d^2Y}{d\bar{x}^2}.$$

Substitute into equation (4.25)

$$\begin{aligned} \varepsilon^2 \cdot \varepsilon^{-2\beta} Y'' - [\varepsilon^\beta \bar{x} q'(x_t) + \mathcal{O}(\varepsilon^{2\beta})] Y = 0 \\ \varepsilon^{2-2\beta} Y'' - \varepsilon^\beta \bar{x} q'(x_t) Y + \mathcal{O}(\varepsilon^{2\beta}) Y = 0. \end{aligned} \tag{4.31}$$

For **dominant balance**, the two leading terms must be of the same order

$$2 - 2\beta = \beta \quad \Rightarrow \quad 3\beta = 2 \quad \Rightarrow \quad \beta = \frac{2}{3}.$$

3.2 Leading Order Equation in Transition Layer

With $\beta = \frac{2}{3}$, equation (4.31) becomes

$$\begin{aligned} \varepsilon^{2-4/3} Y'' - \varepsilon^{2/3} \bar{x} q'(x_t) Y = 0 \\ \varepsilon^{2/3} Y'' - \varepsilon^{2/3} \bar{x} q'(x_t) Y = 0. \end{aligned}$$

Divide by $\varepsilon^{2/3}$

$$Y'' - \bar{x} q'(x_t) Y = 0. \tag{4.33}$$

Let $q_t' = q'(x_t) > 0$, and make the change of variables

$$s = (q_t')^{1/3} \bar{x}.$$

Then

$$\frac{d}{d\bar{x}} = (q_t')^{1/3} \frac{d}{ds}, \quad \frac{d^2}{d\bar{x}^2} = (q_t')^{2/3} \frac{d^2}{ds^2}.$$

Substitute into (4.33)

$$\begin{aligned} (q_t')^{2/3} \frac{d^2Y}{ds^2} - (q_t')^{-1/3} s \cdot q_t' Y = 0 \\ \frac{d^2Y}{ds^2} - sY = 0. \end{aligned} \tag{4.34}$$

This is **Airy's equation**.

3.3 General Solution in Transition Layer

The general solution of Airy's equation is

$$Y_0(s) = a\mathrm{Ai}(s) + b\mathrm{Bi}(s).$$

where Ai and Bi are Airy functions.
In terms of $\bar{x}$

$$Y_0(\bar{x}) = a\mathrm{Ai}[(q_t')^{1/3}\bar{x}] + b\mathrm{Bi}[(q_t')^{1/3}\bar{x}]. \tag{4.35}$$

We expect the solution in the transition layer to have the form

$$Y \sim \varepsilon^{\gamma} Y_0(\bar{x}) + \cdots \tag{4.32}$$

where γ will be determined by matching.

4 Matching Procedure

4.1 Intermediate Variable and Asymptotics

Introduce the intermediate variable

$$x_\eta = \frac{x - x_t}{\varepsilon^\eta}, \quad \text{with } 0 < \eta < \frac{2}{3}. \tag{4.36}$$

Then

$$\bar{x} = \frac{x - x_t}{\varepsilon^{2/3}} = \varepsilon^{\eta - 2/3} x_\eta$$
$$s = (q_t')^{1/3}\bar{x} = (q_t')^{1/3}\varepsilon^{\eta-2/3}x_\eta.$$

Define

$$r = (q_t')^{1/3}\varepsilon^{\eta-2/3}x_\eta.$$

4.2 Asymptotic Expansions of Integrals

For $x > x_t$, we need the asymptotic behavior of

$$\int_{x_t}^{x} \sqrt{q(s)}ds$$

Using the linear approximation $q(s) \sim (s - x_t)q_t'$

$$\int_{x_t}^{x} \sqrt{q(s)}ds \sim \int_{x_t}^{x_t+\varepsilon^\eta x_\eta} \sqrt{(s - x_t)q_t'}ds$$

$$= \sqrt{q_t'}\int_0^{\varepsilon^\eta x_\eta} \sqrt{u}du = \sqrt{q_t'} \cdot \frac{2}{3}(\varepsilon^\eta x_\eta)^{3/2} = \frac{2}{3}\varepsilon^{3\eta/2}x_\eta^{3/2}\sqrt{q_t'} \tag{4.37}$$

Also

$$q(x)^{-1/4} \sim [q_t'(x - x_t)]^{-1/4} = (q_t')^{-1/4}\varepsilon^{-\eta/4}x_\eta^{-1/4} \tag{4.38}$$

4.3 Matching for $x > x_t$

From the transition layer solution, using asymptotic expansions of Airy functions for large positive arguments

$$\mathrm{Ai}(s) \sim \frac{1}{2\sqrt{\pi}s^{1/4}}e^{-\frac{2}{3}s^{3/2}}, \quad \mathrm{Bi}(s) \sim \frac{1}{\sqrt{\pi}s^{1/4}}e^{\frac{2}{3}s^{3/2}} \quad \text{as } s \to +\infty$$

So

$$Y \sim \varepsilon^{\gamma} \left[\frac{a}{2\sqrt{\pi} r^{1/4}} e^{-\frac{2}{3} r^{3/2}} + \frac{b}{\sqrt{\pi} r^{1/4}} e^{\frac{2}{3} r^{3/2}} \right] \tag{4.39}$$

From the WKB solution (4.27)

$$y_R \sim \frac{1}{q(x)^{1/4}} \left[a_R e^{-\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds} + b_R e^{\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds} \right]$$

Using (4.37) and (4.38)

$$y_R \sim \frac{\varepsilon^{-\eta/4}}{(q_t')^{1/4} x_\eta^{1/4}} \left[a_R \exp\left(-\frac{2}{3} \varepsilon^{3\eta/2-1} x_\eta^{3/2} \sqrt{q_t'} \right) + b_R \exp\left(\frac{2}{3} \varepsilon^{3\eta/2-1} x_\eta^{3/2} \sqrt{q_t'} \right) \right]$$

Note that

$$r^{3/2} = (q_t')^{1/2} \varepsilon^{3\eta/2-1} x_\eta^{3/2}$$

so

$$y_R \sim \frac{\varepsilon^{-\eta/4}}{(q_t')^{1/4} x_\eta^{1/4}} \left[a_R e^{-\frac{2}{3} r^{3/2}} + b_R e^{\frac{2}{3} r^{3/2}} \right]$$

But $r^{1/4} = (q_t')^{1/12} \varepsilon^{\eta/4 - 1/6} x_\eta^{1/4}$, so

$$y_R \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6} r^{1/4}} \left[a_R e^{-\frac{2}{3} r^{3/2}} + b_R e^{\frac{2}{3} r^{3/2}} \right] \tag{4.40}$$

Comparing (4.39) and (4.40), we require

$$\varepsilon^{\gamma} = \varepsilon^{-1/6} \quad \Rightarrow \quad \gamma = -\frac{1}{6}$$

and

$$\frac{a}{2\sqrt{\pi}} = \frac{a_R}{(q_t')^{1/6}}, \quad \frac{b}{\sqrt{\pi}} = \frac{b_R}{(q_t')^{1/6}}$$

Thus

$$a_R = \frac{a}{2\sqrt{\pi}} (q_t')^{1/6}, \quad b_R = \frac{b}{\sqrt{\pi}} (q_t')^{1/6} \tag{4.41}$$

4.4 Matching for $x < x_t$

For $x < x_t$, $x_\eta < 0$, so $r < 0$. Let $|r| = -r$.
Using asymptotic expansions of Airy functions for large negative arguments

$$\mathrm{Ai}(s) \sim \frac{1}{\sqrt{\pi} |s|^{1/4}} \sin\left(\frac{2}{3} |s|^{3/2} + \frac{\pi}{4} \right), \quad \mathrm{Bi}(s) \sim \frac{1}{\sqrt{\pi} |s|^{1/4}} \cos\left(\frac{2}{3} |s|^{3/2} + \frac{\pi}{4} \right) \quad \text{as } s \to -\infty$$

So

$$Y \sim \varepsilon^{-1/6} \left[\frac{a}{\sqrt{\pi} |r|^{1/4}} \sin\left(\frac{2}{3} |r|^{3/2} + \frac{\pi}{4} \right) + \frac{b}{\sqrt{\pi} |r|^{1/4}} \cos\left(\frac{2}{3} |r|^{3/2} + \frac{\pi}{4} \right) \right]$$

Using trigonometric identities, this can be rewritten as

$$Y \sim \frac{\varepsilon^{-1/6}}{2\sqrt{\pi} |r|^{1/4}} \left[(a e^{-i\pi/4} + b e^{i\pi/4}) e^{i\zeta} + (a e^{i\pi/4} + b e^{-i\pi/4}) e^{-i\zeta} \right] \tag{4.42}$$

where $\zeta = \frac{2}{3} |r|^{3/2}$.
From the WKB solution (4.28) for $x < x_t$

$$y_L \sim \frac{1}{|q(x)|^{1/4}} \left[a_L e^{i\theta(x)/\varepsilon} + b_L e^{-i\theta(x)/\varepsilon} \right]$$

where $\theta(x) = \int_x^{x_t} \sqrt{|q(s)|}ds$
Similar to before, we find

$$\theta(x) \sim \frac{2}{3}|r|^{3/2} = \zeta$$

and

$$|q(x)|^{-1/4} \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6}|r|^{1/4}}$$

So

$$y_L \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6}|r|^{1/4}}\left[a_L e^{i\zeta} + b_L e^{-i\zeta}\right] \tag{4.43}$$

Comparing (4.42) and (4.43), we get

$$a_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}}(ae^{-i\pi/4} + be^{i\pi/4}), \quad b_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}}(ae^{i\pi/4} + be^{-i\pi/4}) \qquad (4.44, 4.45)$$

5 Connection Formulas and Final Result

From (4.41), we have

$$a = 2\sqrt{\pi}(q_t')^{-1/6}a_R, \quad b = \sqrt{\pi}(q_t')^{-1/6}b_R$$

Substitute into (4.44)

$$a_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}}\left[2\sqrt{\pi}(q_t')^{-1/6}a_R e^{-i\pi/4} + \sqrt{\pi}(q_t')^{-1/6}b_R e^{i\pi/4}\right] = a_R e^{-i\pi/4} + \frac{1}{2}b_R e^{i\pi/4}$$

Similarly for b_L. In matrix form

$$\begin{pmatrix} a_L \\ b_L \end{pmatrix} = \begin{pmatrix} i & \frac{1}{2} \\ 1 & \frac{i}{2} \end{pmatrix}\begin{pmatrix} a_R \\ b_R \end{pmatrix} \tag{4.47}$$

since $e^{\pm i\pi/4} = \frac{1}{\sqrt{2}}(1 \pm i)$ and after normalization.
The final WKB approximation becomes

$$y(x) \sim \begin{cases} \frac{1}{|q(x)|^{1/4}}\left[2a_R \cos\left(\frac{1}{\varepsilon}\theta(x) - \frac{\pi}{4}\right) + b_R \cos\left(\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right)\right] & x < x_t \\ \frac{1}{q(x)^{1/4}}\left(a_R e^{-k(x)/\varepsilon} + b_R e^{k(x)/\varepsilon}\right) & x > x_t \end{cases} \tag{4.48}$$

where

$$\theta(x) = \int_x^{x_t} \sqrt{|q(s)|}ds, \quad k(x) = \int_{x_t}^x \sqrt{q(s)}ds$$

6 Uniform Approximation (Langer, 1931)

A single composite expansion valid through the turning point

$$y(x) \sim \varepsilon^{-1/6}\left(\frac{f(x)}{q(x)}\right)^{1/4}\left[a_0 \mathrm{Ai}(\varepsilon^{-2/3}f(x)) + b_0 \mathrm{Bi}(\varepsilon^{-2/3}f(x))\right] \tag{4.63}$$

where

$$f(x) = \begin{cases} \left[\frac{3}{2}\int_{x_t}^x \sqrt{q(s)}ds\right]^{2/3} & x \geq x_t \\ -\left[\frac{3}{2}\int_x^{x_t} \sqrt{-q(s)}ds\right]^{2/3} & x \leq x_t \end{cases}$$

The connection with the WKB constants is

$$a_R = \frac{a_0}{2\sqrt{\pi}}, \quad b_R = \frac{b_0}{\sqrt{\pi}}$$

This completes the detailed derivation of the WKB method for turning points.