The Method of Multiple Scales

1 Introduction

In the previous lecture, we studied **Matched Asymptotic Expansions** (MAE), which constructs solutions in different spatial regions (e.g., boundary layers) and matches them together. The **Method of Multiple Scales** (MMS) differs fundamentally:

- **MAE:** Patching separate expansions from different spatial regions
- **MMS:** Starting with a generalized composite expansion using multiple *time scales* treated as independent variables

This approach transforms ordinary differential equations (ODEs) into partial differential equations (PDEs), which paradoxically makes them easier to solve for long-time behavior.

Historical Context

The method originated in celestial mechanics:

- **Stokes (1843):** Early use of coordinate expansions in fluid flow
- **Lindstedt (1882), Poincaré (1886):** Systematic removal of **secular terms** (unbounded terms that ruin long-time approximations)
- **Modern formulation:** Kuzmak (1959), Cole & Kevorkian (1963) introduced multiple independent variables based on the expansion parameter

2 Introductory Example: Weakly Damped Oscillator

Consider the classic problem:

$$y'' + \varepsilon y' + y = 0, \quad t > 0$$

with initial conditions:

$$y(0) = 0, \quad y'(0) = 1$$

where $0 < \varepsilon \ll 1$. This models a weakly damped oscillator.

2.1 Exact Solution

The characteristic equation is:

$$r^2 + \varepsilon r + 1 = 0$$

with roots:

$$r = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2} = -\frac{\varepsilon}{2} \pm i\sqrt{1 - \frac{\varepsilon^2}{4}}$$

Thus the exact solution is:

$$y(t) = e^{-\varepsilon t/2}\left[C_1 \cos\left(t\sqrt{1-\frac{\varepsilon^2}{4}}\right) + C_2 \sin\left(t\sqrt{1-\frac{\varepsilon^2}{4}}\right)\right]$$

Applying initial conditions $y(0) = 0$, $y'(0) = 1$:

$$y(0) = C_1 = 0$$

$$y'(t) = e^{-\varepsilon t/2}\left[-\frac{\varepsilon}{2}C_2\sin(\omega t) + \omega C_2\cos(\omega t)\right] - \frac{\varepsilon}{2}e^{-\varepsilon t/2}C_2\sin(\omega t)$$

where $\omega = \sqrt{1-\varepsilon^2/4}$. Then:

$$y'(0) = \omega C_2 = 1 \Rightarrow C_2 = \frac{1}{\omega}$$

Therefore, the exact solution is:

$$y(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}}e^{-\varepsilon t/2}\sin\left(t\sqrt{1-\varepsilon^2/4}\right) \tag{2.1}$$

2.2 Regular Expansion Failure

A regular perturbation expansion $y \sim y_0(t) + \varepsilon y_1(t) + \cdots$ yields:

$$y \sim \sin t - \tfrac{1}{2}\varepsilon t\sin t + O(\varepsilon^2)$$

The term $-\frac{1}{2}\varepsilon t\sin t$ is **secular**—grows unbounded, making the approximation invalid for $t \sim O(1/\varepsilon)$.

2.3 Multiple-Scale Expansion with Undetermined Scaling

Step 1: Introduce Multiple Scales with Undetermined Exponent

$$t_1 = t \quad \text{(fast time)}, \qquad t_2 = \varepsilon^\alpha t \quad \text{(slow time, } \alpha \text{ to be determined)}$$

Treat t_1 and t_2 as independent variables.

Step 2: Transform Derivatives

$$\frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon^\alpha\frac{\partial}{\partial t_2}, \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_1^2} + 2\varepsilon^\alpha\frac{\partial^2}{\partial t_1\partial t_2} + \varepsilon^{2\alpha}\frac{\partial^2}{\partial t_2^2}$$

The ODE becomes:

$$(\partial_{t_1}^2 + 2\varepsilon^\alpha\partial_{t_1}\partial_{t_2} + \varepsilon^{2\alpha}\partial_{t_2}^2)y + \varepsilon(\partial_{t_1} + \varepsilon^\alpha\partial_{t_2})y + y = 0 \tag{3}$$

Step 3: Assume Perturbation Expansion

$$y \sim y_0(t_1,t_2) + \varepsilon y_1(t_1,t_2) + \varepsilon^2 y_2(t_1,t_2) + \cdots$$

Step 4: Solve Order-by-Order

$O(1)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_0 = 0$$

General solution:

$$y_0 = A_0(t_2)\sin t_1 + B_0(t_2)\cos t_1 \tag{4}$$

Coefficients A_0, B_0 depend on slow scale t_2.

$O(\varepsilon)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_1 = -2\varepsilon^{\alpha-1}\partial_{t_1}\partial_{t_2}y_0 - \partial_{t_1}y_0 \tag{5}$$

Step 5: Determine α and Remove Secular Terms

Substitute (4) into (5):

$$(\partial_{t_1}^2 + 1)y_1 = -(2\varepsilon^{\alpha-1}A_0' + A_0)\cos t_1 + (2\varepsilon^{\alpha-1}B_0' + B_0)\sin t_1$$

To remove secular terms, we must balance the coefficients. The only choice is $\alpha = 1$, which gives:

$$(\partial_{t_1}^2 + 1)y_1 = -(2A_0' + A_0)\cos t_1 + (2B_0' + B_0)\sin t_1$$

Remove secular terms by setting coefficients to zero:

$$2A_0' + A_0 = 0 \quad \text{and} \quad 2B_0' + B_0 = 0$$

Step 6: Solve Slow Evolution Equations

With initial conditions $y_0(0,0) = 0$, $\partial_{t_1} y_0(0,0) = 1$:

$$A_0(t_2) = e^{-t_2/2}, \quad B_0(t_2) = 0$$

Step 7: Final Uniform Approximation

$$y \sim y_0 = e^{-t_2/2} \sin t_1 = e^{-\varepsilon t/2} \sin t \tag{6}$$

This approximation:

- Captures both oscillation and decay
- Is valid for $t = O(1/\varepsilon)$
- Matches the exact solution remarkably well

3 Labor-Saving Techniques

3.1 Efficient Solution Forms

For the $O(1)$ problem $y'' + \omega^2 y = 0$, use:

- $y = A\cos(\omega t + \theta)$ or $y = Re^{i\omega t} + \bar{R}e^{-i\omega t}$
- These forms make secular term identification easier, especially for nonlinear problems

3.2 Direct Secularity Condition

For $y'' + \omega^2 y = f(t)$, secular terms arise if $f(t)$ contains $\sin(\omega t)$ or $\cos(\omega t)$.

Shortcut: Instead of solving the $O(\varepsilon)$ equation completely, set coefficients of $\sin t_1$ and $\cos t_1$ in (5) to zero. This directly gives the slow evolution equations for A_0 and B_0.

4 Extensions and Refinements

4.1 Three Time Scales: Detailed Derivation

To obtain an approximation valid for longer times ($t = O(1/\varepsilon^2)$), we introduce three time scales:

$$t_1 = t, \quad t_2 = \varepsilon t, \quad t_3 = \varepsilon^2 t$$

Derivative Transformations

The time derivative becomes:

$$\frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2} + \varepsilon^2 \frac{\partial}{\partial t_3}$$

For the second derivative:

$$\begin{aligned} \frac{d^2}{dt^2} &= \left(\frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2} + \varepsilon^2 \frac{\partial}{\partial t_3} \right)^2 \\ &= \partial_{t_1}^2 + 2\varepsilon \partial_{t_1}\partial_{t_2} + \varepsilon^2 \left(\partial_{t_2}^2 + 2\partial_{t_1}\partial_{t_3} \right) + O(\varepsilon^3) \end{aligned}$$

Transformed Equation

Substituting into our model equation $y'' + \varepsilon y' + y = 0$:

$$\left[\partial_{t_1}^2 + 2\varepsilon \partial_{t_1}\partial_{t_2} + \varepsilon^2 \left(\partial_{t_2}^2 + 2\partial_{t_1}\partial_{t_3} \right) \right] y + \varepsilon \left(\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} \right) y + y = 0$$

Perturbation Expansion

Assume:

$$y \sim y_0(t_1, t_2, t_3) + \varepsilon y_1(t_1, t_2, t_3) + \varepsilon^2 y_2(t_1, t_2, t_3) + \cdots$$

$O(1)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_0 = 0$$

General solution:

$$y_0 = A_0(t_2, t_3)\sin t_1 + B_0(t_2, t_3)\cos t_1 \tag{7.1}$$

$O(\varepsilon)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_1 = -2\partial_{t_1}\partial_{t_2} y_0 - \partial_{t_1} y_0$$

Substitute (7.1):

$$\begin{aligned} -2\partial_{t_1}\partial_{t_2} y_0 &= -2(A_0' \cos t_1 - B_0' \sin t_1) \\ -\partial_{t_1} y_0 &= -(A_0 \cos t_1 - B_0 \sin t_1) \end{aligned}$$

where $(') \equiv \partial/\partial t_2$.
Thus:

$$(\partial_{t_1}^2 + 1)y_1 = -(2A_0' + A_0)\cos t_1 + (2B_0' + B_0)\sin t_1$$

Remove secular terms:

$$2A_0' + A_0 = 0, \quad 2B_0' + B_0 = 0$$

Solutions:

$$A_0(t_2, t_3) = \alpha_0(t_3)e^{-t_2/2}, \quad B_0(t_2, t_3) = \beta_0(t_3)e^{-t_2/2} \tag{7.2}$$

From initial conditions $y_0(0,0,0) = 0$, $\partial_{t_1} y_0(0,0,0) = 1$:

$$\alpha_0(0) = 1, \quad \beta_0(0) = 0$$

$O(\varepsilon^2)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_2 = -2\partial_{t_1}\partial_{t_2} y_1 - (\partial_{t_2}^2 + 2\partial_{t_1}\partial_{t_3})y_0 - \partial_{t_1} y_1 - \partial_{t_2} y_0$$

After detailed computation, the secularity conditions become:

$$2\partial_{t_2} A_1 + A_1 = \tfrac{1}{2}(\beta_0 - 8\alpha_0')e^{-t_2/2} \tag{7.3}$$

$$2\partial_{t_2} B_1 + B_1 = -\tfrac{1}{2}(\alpha_0 + 8\beta_0')e^{-t_2/2} \tag{7.4}$$

where $(') \equiv \partial/\partial t_3$ for α_0, β_0.
To minimize the contribution of y_1, we set the right-hand sides to zero:

$$\beta_0 - 8\alpha_0' = 0, \quad \alpha_0 + 8\beta_0' = 0$$

Differentiate the first equation and substitute:

$$\alpha_0'' + \tfrac{1}{64}\alpha_0 = 0$$

Solution: $\alpha_0(t_3) = \cos(t_3/8)$, $\beta_0(t_3) = -\sin(t_3/8)$

Final Three-Scale Approximation

Substitute into (7.1) and (7.2):

$$y \sim e^{-t_2/2}\left[\cos(t_3/8)\sin t_1 - \sin(t_3/8)\cos t_1\right] = e^{-\varepsilon t/2}\sin\left(t - \tfrac{1}{8}\varepsilon^2 t\right)$$

Thus:

$$y \sim e^{-\varepsilon t/2}\sin(\omega t), \quad \omega = 1 - \tfrac{1}{8}\varepsilon^2 \tag{7.5}$$

4.2 Two-Term Expansion: Detailed Derivation

Using only two scales ($t_1 = t$, $t_2 = \varepsilon t$) but including the second term in the expansion.

Perturbation Expansion

Assume:

$$y \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \varepsilon^2 y_2(t_1, t_2)$$

We already have from the two-scale analysis:

$$y_0 = e^{-t_2/2} \sin t_1$$

$O(\varepsilon)$ Solution

The general solution for y_1 is:

$$y_1 = A_1(t_2) \sin t_1 + B_1(t_2) \cos t_1 - \tfrac{1}{2}(2B_0' + B_0)t_1 \cos t_1 - \tfrac{1}{2}(2A_0' + A_0)t_1 \sin t_1$$

But from earlier, $2A_0' + A_0 = 0$ and $2B_0' + B_0 = 0$, so the secular terms vanish:

$$y_1 = A_1(t_2) \sin t_1 + B_1(t_2) \cos t_1$$

$O(\varepsilon^2)$ Problem and Secularity Conditions

At $O(\varepsilon^2)$:

$$(\partial_{t_1}^2 + 1)y_2 = -2\partial_{t_1}\partial_{t_2} y_1 - \partial_{t_2}^2 y_0 - \partial_{t_1} y_1 - \partial_{t_2} y_0$$

After substituting known expressions and extracting secular terms, we find:

$$2A_1' + A_1 = 0 \tag{7.6}$$

$$2B_1' + B_1 = -\tfrac{1}{2} e^{-t_2/2} \tag{7.7}$$

Solving for A_1 and B_1

From (7.6): $A_1(t_2) = a_1 e^{-t_2/2}$

From (7.7): This is a first-order linear ODE. The homogeneous solution is $B_1^{(h)} = b_1 e^{-t_2/2}$. A particular solution is $B_1^{(p)} = -\frac{1}{2} t_2 e^{-t_2/2}$.
Thus:

$$B_1(t_2) = b_1 e^{-t_2/2} - \tfrac{1}{2} t_2 e^{-t_2/2}$$

Initial Conditions

From $y(0) = 0$:

$$y_0(0,0) + \varepsilon y_1(0,0) = 0 \Rightarrow B_1(0) = 0 \Rightarrow b_1 = 0$$

From $y'(0) = 1$:

$$\partial_{t_1} y_0(0,0) + \varepsilon(\partial_{t_2} y_0(0,0) + \partial_{t_1} y_1(0,0)) = 1$$

We have $\partial_{t_1} y_0(0,0) = 1$, $\partial_{t_2} y_0(0,0) = 0$, $\partial_{t_1} y_1(0,0) = A_1(0)$
Thus: $1 + \varepsilon A_1(0) = 1 \Rightarrow A_1(0) = 0 \Rightarrow a_1 = 0$

Final Two-Term Expansion

$$y \sim e^{-t_2/2} \sin t_1 - \tfrac{1}{2}\varepsilon t_2 e^{-t_2/2} \cos t_1$$

In original variables:

$$y \sim e^{-\varepsilon t/2} \left(\sin t - \tfrac{1}{2}\varepsilon^2 t \cos t \right) \tag{7.8}$$

4.3 Comparison of Approximations

We now have three different approximations:

(i) **One-term, two scales:** $y \sim e^{-\varepsilon t/2} \sin t$

(ii) **Two-term, two scales:** $y \sim e^{-\varepsilon t/2}(\sin t - \frac{1}{2}\varepsilon^2 t \cos t)$

(iii) **One-term, three scales:** $y \sim e^{-\varepsilon t/2} \sin(\omega t)$, $\omega = 1 - \frac{1}{8}\varepsilon^2$

- **(i) vs (ii):** For $t = O(1/\varepsilon)$, (ii) is more accurate as it includes the next-order correction
- **(i) vs (iii):** (iii) remains valid for longer times $t = O(1/\varepsilon^2)$ due to corrected frequency
- **(ii) vs (iii):** (ii) is more accurate for moderate times, but (iii) maintains accuracy for longer durations
- **Effort:** Finding (ii) and (iii) requires similar computational effort, as both need analysis up to $O(\varepsilon^2)$

5 The Minimum Error Principle

The multiple-scale expansion is not unique. Different choices can satisfy the no-secular-term condition. The preferred expansion is chosen using the **principle of minimum error**:

Among all possible asymptotic expansions satisfying solvability conditions, choose the one that minimizes the maximum error between the approximation and the true solution over the time interval of interest.

This principle helps determine otherwise arbitrary functions that appear in higher-order analyses.

6 General Methodology

The Method of Multiple Scales follows these steps:

1. **Identify relevant scales** based on physical insight or dominant balance
2. **Treat scales as independent variables**, transforming ODEs to PDEs
3. **Assume a perturbation expansion** in the multi-variable function
4. **Solve order-by-order**, obtaining general solutions with slow-varying coefficients
5. **Remove secular terms** at each order by imposing solvability conditions
6. **Recombine** by replacing scaled variables with original variables

Common Time Scales

- **Two-timing:** $t_1 = t$, $t_2 = \varepsilon t$
- **Multiple scales:** $t_1 = t$, $t_2 = \varepsilon t$, $t_3 = \varepsilon^2 t$, ...
- **Strained coordinates:** $t_1 = (1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots)t$, $t_2 = \varepsilon t$
- **Nonlinear time:** $t_1 = f(t, \varepsilon)$, $t_2 = \varepsilon t_1$

7 Summary

The Method of Multiple Scales is essential for:

- Problems with solutions exhibiting behavior on vastly different time scales
- Removing secular terms and obtaining uniformly valid approximations
- Studying long-time behavior of oscillatory systems with weak nonlinearities or damping
- Applications in nonlinear oscillations, wave propagation, and many other areas

The key insight is that introducing multiple independent scales and using the freedom in slow-varying coefficients to remove secular terms leads to approximations that remain valid for long times.