Matched Asymptotic Expansions: Boundary, Interior, and Corner Layers

In this lecture we generalize the matched asymptotic expansion method introduced in Sect. 2.2. Almost all essential ideas were already illustrated by the boundary-layer example, so our goal here is to recognize how those ideas apply to a broad class of problems and how new phenomena—especially exponentially small and interior-layer effects—arise.

1 General Features of Matched Asymptotic Expansions

Consider a singularly perturbed boundary value problem that can be written schematically as

$$L_0[y] + \varepsilon L_1[y] + \varepsilon^2 L_2[y] + \cdots = 0, \qquad a < x < b,$$

where $L_0, L_1, L_2, \ldots$ are differential operators and L_k may contain derivatives up to some order. Typically L_2 (and higher) contain the highest derivatives, multiplied by small parameters.

Setting $\varepsilon = 0$ gives the **reduced (outer) problem**

$$L_0[y_0] = 0,$$

which usually has lower differential order and therefore cannot satisfy all boundary conditions. This loss of boundary information signals a singular perturbation.

Essential Structure

(i) **Outer region:** Solve $L_0[y_0] = 0$ on $a < x < b$, applying only those boundary conditions that do not force rapid variation.

(ii) **Inner region(s):** Near each layer point $x = x_i$, introduce a stretched coordinate

$$\bar{x}_i = \frac{x - x_i}{\varepsilon^{\alpha_i}},$$

and derive a leading inner equation by retaining the terms in $L_0, L_1, L_2, \ldots$ that are of comparable size under that scaling.

(iii) **Matching:** Impose that the inner solution (as $\bar{x}_i \to \pm\infty$) matches the outer solution (as $x \to x_i$).

(iv) **Composite expansion:** Form a uniformly valid approximation

$$y_{\text{comp}} = y_{\text{outer}} + y_{\text{inner}} - y_{\text{common}},$$

where y_{common} is the part counted twice.

Remarks

- The power α_i in the stretched coordinate is chosen by a dominant-balance argument so that at least two terms in the equation survive at leading order.
- Multiple layers can appear: at both ends, at interior points, or even nested at a single point with different thicknesses.
- The composite expansion is not unique algebraically; what matters is that it is uniformly accurate.

1.1 Example 1: Two Boundary Layers (Holmes §2.3)

We study

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^x, \qquad 0 < x < 1, \qquad y(0) = 2,\ y(1) = 1.$$

We rewrite this in the hierarchical operator form

$$L_0[y] + \varepsilon L_1[y] + \varepsilon^2 L_2[y] = 0,$$

with

$$L_0[y] = -y + e^x, \qquad L_1[y] = x\,y', \qquad L_2[y] = y''.$$

Outer problem. Set $\varepsilon = 0$ to obtain

$$L_0[y_0] = 0 \quad \Longrightarrow \quad -y_0 + e^x = 0 \quad \Longrightarrow \quad y_0(x) = e^x.$$

This solves the reduced equation, but it does *not* satisfy either boundary condition $y(0) = 2$, $y(1) = 1$, so boundary layers must appear at both $x = 0$ and $x = 1$.

Left inner layer near $x = 0$. Introduce the stretched coordinate

$$\bar{x} = \frac{x}{\varepsilon^\alpha}, \qquad y(x) = Y(\bar{x}).$$

Then

$$\frac{dy}{dx} = \frac{1}{\varepsilon^\alpha} Y', \qquad \frac{d^2 y}{dx^2} = \frac{1}{\varepsilon^{2\alpha}} Y'',$$

so the differential equation

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^x$$

becomes

$$\varepsilon^{2-2\alpha} Y'' \ + \ \varepsilon \bar{x}\, Y' \ - \ Y \ = \ -e^{\varepsilon \bar{x}}.$$

In the $\varepsilon \to 0$ limit with $\bar{x} = O(1)$, we have $e^{\varepsilon \bar{x}} = 1 + O(\varepsilon)$ and $\bar{x} = O(1)$. We now choose α so that the leading-order balance in this inner region is nontrivial and can enforce the boundary condition at $x = 0$.

- If $\alpha < 1$, then $\varepsilon^{2-2\alpha} \to 0$ even faster, so the Y'' term is too small to influence the leading-order balance. The leading-order equation would become

$$-Y = -1,$$

i.e. $Y \equiv 1$, which cannot satisfy $y(0) = 2$. So $\alpha < 1$ cannot generate enough curvature to fix the boundary condition.

- If $\alpha > 1$, then $2 - 2\alpha < 0$, so $\varepsilon^{2-2\alpha}$ blows up and the term $\varepsilon^{2-2\alpha} Y''$ is the leading term. We then have

$$\varepsilon^{2-2\alpha} Y'' = 0 \quad \Rightarrow \quad Y_0'' = 0.$$

Thus Y_0 would have to be linear on this scale:

$$Y_0(\bar{x}) = a\bar{x} + 2.$$

But this does not work: as $\bar{x} \to \infty$ we would need $Y_0(\bar{x}) \to y_0(0) = 1$, which is a constant limit, while a nontrivial linear function cannot tend to a constant. If we force $a = 0$ to make it constant, then $Y_0 \equiv 2$, which cannot match to $y_0(0) = 1$. So with $\alpha > 1$ we cannot match $Y_0(\infty)$ to the outer solution, nor can we repair this in a thin transition region. This scaling fails.

- If $\alpha = 1$, then $2 - 2\alpha = 0$, so $\varepsilon^{2-2\alpha} = \varepsilon^0 = 1$. The equation becomes

$$Y'' + \varepsilon \bar{x}\, Y' - Y = -e^{\varepsilon \bar{x}}.$$

Now Y'' and Y both enter at $O(1)$, and the $\varepsilon \bar{x} Y'$ term is smaller (order ε). Meanwhile the right-hand side is $-1 + O(\varepsilon)$. This gives a nontrivial leading-order balance between Y'' and Y, which *can* adjust the boundary value.

Therefore the distinguished scaling is

$$\alpha = 1, \qquad \bar{x} = \frac{x}{\varepsilon}.$$

With $\alpha = 1$, the leading inner equation is

$$Y_0'' - Y_0 = -1, \qquad Y_0(0) = 2.$$

This ODE solves to

$$Y_0(\bar{x}) = 1 + Ae^{-\bar{x}} + Be^{\bar{x}}.$$

Matching as $\bar{x} \to \infty$ to the outer limit $y_0(0^+) = 1$ requires $B = 0$ and then $A = 1$ from $Y_0(0) = 2$, so

$$Y_0(\bar{x}) = 1 + e^{-\bar{x}}.$$

Right inner layer near $x = 1$. We now look for a boundary layer near $x = 1$ that can satisfy $y(1) = 1$. Introduce the stretched coordinate

$$\tilde{x} = \frac{x-1}{\varepsilon^\beta}, \qquad y(x) = \widetilde{Y}(\tilde{x}).$$

Then, using the chain rule,

$$\frac{dy}{dx} = \frac{1}{\varepsilon^\beta}\widetilde{Y}', \qquad \frac{d^2y}{dx^2} = \frac{1}{\varepsilon^{2\beta}}\widetilde{Y}''.$$

Substitute these into the differential equation

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^x$$

to get

$$\varepsilon^{2-2\beta}\widetilde{Y}'' \;+\; \varepsilon^{1-\beta}\big(1 + \varepsilon^\beta \tilde{x}\big)\widetilde{Y}' \;-\; \widetilde{Y} \;=\; -e^{\,1+\varepsilon^\beta \tilde{x}}.$$

We again choose β to get a nontrivial leading-order balance as $\varepsilon \to 0$ with $\tilde{x} = O(1)$.

- If $\beta < 1$, then $2 - 2\beta > 0$ and $1 - \beta > 0$, so

$$\varepsilon^{2-2\beta} \to 0, \qquad \varepsilon^{1-\beta} \to 0.$$

Both derivative terms drop out at leading order. We would be left with

$$\widetilde{Y}_0 \equiv e.$$

But this constant value e does *not* satisfy the boundary condition $y(1) = 1$. So $\beta < 1$ cannot produce the needed correction at $x = 1$.

- If $\beta > 1$, then $2 - 2\beta < 0$, forcing $\widetilde{Y}_0'' = 0$. Then again $\widetilde{Y}_0$ would be linear and $\widetilde{Y}_0 = a\tilde{x} + 1$, which cannot match the outer limit $y_0(1^-) = e$. So $\beta > 1$ also fails to give a viable matching.
- If $\beta = 1$, then we obtain

$$\widetilde{Y}'' + (1 + \varepsilon\tilde{x})\widetilde{Y}' - \widetilde{Y} \;=\; -e^{1+\varepsilon\tilde{x}},$$

yielding at the leading order

$$\widetilde{Y}_0'' + \widetilde{Y}_0' - \widetilde{Y}_0 = -e, \qquad \widetilde{Y}_0(0) = 1.$$

The homogeneous equation

$$\widetilde{Y}_0'' + \widetilde{Y}_0' - \widetilde{Y}_0 = 0$$

has characteristic equation

$$r^2 + r - 1 = 0 \quad \Rightarrow \quad r_\pm = \frac{-1 \pm \sqrt{5}}{2}.$$

A constant particular solution to the inhomogeneous equation is $\widetilde{Y}_p = e$, so the general leading-order inner solution is

$$\widetilde{Y}_0(\tilde{x}) = e + C_+ e^{r_+\tilde{x}} + C_- e^{r_-\tilde{x}}.$$

Next: we impose matching and boundedness.

As $\tilde{x} \to -\infty$ (which corresponds to $x \to 1^-$ from inside the domain), the solution should approach the outer solution value

$$y_0(1^-) = e^1 = e,$$

and remain bounded. Because $r_+ > 0$, the term $e^{r_+\tilde{x}}$ would *blow up* as $\tilde{x} \to -\infty$, so we must set $C_+ = 0$.
Finally, we use the boundary condition at $x = 1$, i.e. at $\tilde{x} = 0$:

$$\widetilde{Y}_0(0) = e + C_- = 1 \quad \Longrightarrow \quad C_- = 1 - e.$$

This gives the correct boundary value $y(1) = 1$ at $\tilde{x} = 0$, and smoothly relaxes back to the outer state e as $\tilde{x} \to -\infty$.

Composite Expansion.

$$y \sim y_0(x) + Y_0(\bar{x}) - Y_0(\infty) + \tilde{Y}_0(\tilde{x}) - \tilde{Y}_0(-\infty).$$

1.2 Example 4: Multiple Nested Layers Near a Single Boundary (Holmes §2.3)

Consider the problem

$$\varepsilon^3 y'' + x^3 y' - \varepsilon y = x^3, \qquad 0 < x < 1,$$

with boundary conditions

$$y(0) = 1, \qquad y(1) = 3.$$

This problem develops *two nested boundary layers near* $x = 0$: an extremely thin one of thickness $O(\varepsilon)$ and a thicker one of thickness $O(\varepsilon^{1/2})$. Together they correct the outer solution so that both boundary conditions are satisfied.

Outer solution. Set $\varepsilon = 0$ in the differential equation. We obtain

$$x^3 y_0' = x^3 \quad \Longrightarrow \quad y_0' = 1 \quad \Longrightarrow \quad y_0(x) = x + C.$$

Imposing $y(1) = 3$ gives $3 = 1 + C$, so $C = 2$ and

$$y_0(x) = x + 2 \quad \text{or} \quad y_0(x) = x + 1.$$

Quickly check that working with the latter outer solution fails at some stage. The foregoing outer solution satisfies the boundary condition at $x = 1$. However, $y_0(0) = 2 \neq 1$, so the outer solution fails to satisfy $y(0) = 1$. Therefore we need corrections near $x = 0$. The key fact is that one correction layer is *not* enough; we will find two distinct inner scalings.

General inner scaling near $x = 0$

Introduce a stretched coordinate

$$\bar{x} = \frac{x}{\varepsilon^\alpha}, \qquad y(x) = Y(\bar{x}),$$

with $\alpha > 0$ to be determined.
Then

$$\frac{dy}{dx} = \frac{1}{\varepsilon^\alpha} Y', \qquad \frac{d^2y}{dx^2} = \frac{1}{\varepsilon^{2\alpha}} Y'', \qquad x = \varepsilon^\alpha \bar{x}, \qquad x^3 = \varepsilon^{3\alpha}\bar{x}^3.$$

Substitute into the original ODE:

$$\varepsilon^3 \left(\frac{1}{\varepsilon^{2\alpha}} Y''\right) + x^3 \left(\frac{1}{\varepsilon^\alpha} Y'\right) - \varepsilon Y = x^3,$$

which becomes

$$\varepsilon^{3-2\alpha} Y'' + \varepsilon^{2\alpha}\bar{x}^3 Y' - \varepsilon Y = \varepsilon^{3\alpha}\bar{x}^3. \tag{$\star$}$$

To identify relevant layer thicknesses, we look for values of α for which at least two of these terms balance at leading order as $\varepsilon \to 0$ with $\bar{x} = O(1)$.

Two distinguished balances occur:

- **Balance 1:** match $\varepsilon^{3-2\alpha} Y''$ against $-\varepsilon Y$. Requiring $3 - 2\alpha = 1$ gives $\alpha = 1$. This predicts an *inner–inner layer* of thickness $x = O(\varepsilon)$.

- **Balance 2:** match $\varepsilon^{2\alpha}\bar{x}^3 Y'$ against $-\varepsilon Y$. Requiring $2\alpha = 1$ gives $\alpha = \frac{1}{2}$. This predicts an *inner layer* of thickness $x = O(\varepsilon^{1/2})$.

Thus, unlike standard single-layer problems, this example contains *two nested layers at the same boundary* $x = 0$:

$$x = O(\varepsilon) \quad \text{inside} \quad x = O(\varepsilon^{1/2}).$$

We now solve each layer.

Inner–inner layer: $x = O(\varepsilon)$

Take

$$\bar{x} = \frac{x}{\varepsilon}, \qquad y(x) = Y(\bar{x}) \quad (\alpha = 1).$$

Then

$$\varepsilon^{3-2\alpha} = \varepsilon^1, \quad \varepsilon^{2\alpha} = \varepsilon^2, \quad \varepsilon^{3\alpha} = \varepsilon^3.$$

Equation $(\star)$ becomes

$$\varepsilon Y'' + \varepsilon^2 \bar{x}^3 Y' - \varepsilon Y = \varepsilon^3 \bar{x}^3.$$

The leading-order inner–inner equation is

$$Y_0'' - Y_0 = 0, \qquad Y_0(0) = y(0) = 1.$$

The general solution is

$$Y_0(\bar{x}) = Ae^{\bar{x}} + Be^{-\bar{x}} = Ae^{\frac{x}{\varepsilon}} + Be^{-\frac{x}{\varepsilon}}.$$

Inner layer: $x = O(\varepsilon^{1/2})$

Now take

$$X = \frac{x}{\varepsilon^{1/2}}, \qquad y(x) = W(X) \quad (\alpha = \tfrac{1}{2}).$$

Then

$$\frac{dy}{dx} = \frac{1}{\varepsilon^{1/2}} W', \qquad \frac{d^2y}{dx^2} = \frac{1}{\varepsilon} W'', \qquad x = \varepsilon^{1/2} X, \qquad x^3 = \varepsilon^{3/2} X^3.$$

Substitute into the ODE:

$$\varepsilon^3 \left(\frac{1}{\varepsilon} W'' \right) + x^3 \left(\frac{1}{\varepsilon^{1/2}} W' \right) - \varepsilon W = x^3,$$

i.e.

$$\varepsilon^2 W'' + \varepsilon X^3 W' - \varepsilon W = \varepsilon^{3/2} X^3,$$

yielding

$$X^3 W_0' - W_0 = 0.$$

This is first order and separable:

$$\frac{W_0'}{W_0} = \frac{1}{X^3} \quad \Longrightarrow \quad \ln W_0 = \int \frac{1}{X^3}\, dX,$$

so

$$W_0(X) = Ce^{-\frac{1}{2X^2}} = Ce^{-\frac{\varepsilon}{2x^2}}.$$

We now determine A, B, and C by matching the boundary condition at $x = 0$ and the outer solution as $x \to 0^+$.

From the $x = O(\varepsilon)$ layer, we wrote

$$Y_0(\bar{x}) = Ae^{\bar{x}} + Be^{-\bar{x}}, \qquad \bar{x} = \frac{x}{\varepsilon}.$$

For fixed $x > 0$ (i.e. outside the $O(\varepsilon)$ layer), $\bar{x} \to \infty$, and $e^{\bar{x}}$ would blow up unless $A = 0$, so we must set $A = 0$. Thus

$$Y_0(\bar{x}) = Be^{-\bar{x}} = Be^{-x/\varepsilon}.$$

This term becomes exponentially small for any fixed $x > 0$, so it cannot match the outer solution directly. Its only job is to satisfy the boundary condition at $x = 0$. Enforcing $y(0) = 1$ gives $Y_0(0) = B = 1$.

From the $x = O(\varepsilon^{1/2})$ layer, we found

$$W_0(X) = Ce^{-\frac{1}{2X^2}} = Ce^{-\frac{\varepsilon}{2x^2}}, \qquad \left(X = \frac{x}{\varepsilon^{1/2}}\right).$$

As $X \to 0^+$ (i.e. $x \ll \varepsilon^{1/2}$), we have $W_0 \to 0$, so this layer does *not* enforce $y(0) = 1$. As $X \to \infty$ (i.e. $x \to 0^+$ but $x \gg \varepsilon^{1/2}$), we get $\exp\bigl(-1/(2X^2)\bigr) \to 1$, so $W_0 \to C$. In that same limit, the outer solution satisfies

$$y_0(x) = x + 2 \quad \Longrightarrow \quad y_0(0^+) = 2.$$

Therefore we must take $C = 2$ to match smoothly onto the outer solution.

Putting everything together: - outer solution: $y_0(x) = x + 2$, - $O(\varepsilon)$ (inner–inner) layer: $e^{-x/\varepsilon}$, - $O(\varepsilon^{1/2})$ (inner) layer: $2e^{-\frac{\varepsilon}{2x^2}}$, we obtain the composite first approximation

$$y(x) \sim x + 2 \; + \; e^{-x/\varepsilon} \; + \; 2e^{-\frac{\varepsilon}{2x^2}} - 2.$$

2 Transcendentally Small Terms (Holmes §2.4)

Consider the BVP

$$\varepsilon y'' = 2 - y', \quad y(0) = 0, \; y(1) = 1$$

with exact solution

$$y = 2x - \frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}$$

Outer: $y_0 = 2x - 1$, $0 = y_1 = y_2 = \cdots = y_n = \cdots$. Need transcendentally small terms in the outer expansions.

3 Interior Layers (Holmes §2.5)

Consider

$$\varepsilon y'' = y\,y' - y, \qquad 0 < x < 1, \qquad y(0) = 1, \; y(1) = -1.$$

Outer problem and multiple branches

Set $\varepsilon = 0$. The reduced (outer) equation is

$$y_0\,y_0' - y_0 = 0 \quad \Longrightarrow \quad y_0(y_0' - 1) = 0.$$

This factorization gives *two possible outer branches*:

$$\text{Branch A: } y_0 \equiv 0, \qquad \text{Branch B: } y_0' = 1 \;\Rightarrow\; y_0 = x + a.$$

These are both $O(1)$ solutions on subintervals of $(0, 1)$.

However, no single branch can satisfy both boundary conditions:

- Branch A ($y_0 \equiv 0$) clearly cannot satisfy $y(0) = 1$ or $y(1) = -1$.
- Branch B ($y_0 = x + a$) can match the left boundary by choosing $a = 1$ so that $y_0(0) = 1$, but then $y_0(1) = 2 \neq -1$. Alternatively, it can match the right boundary by choosing $a = -2$ so that $y_0(1) = -1$, but then $y_0(0) = -2 \neq 1$.

Conclusion: the boundary conditions are "asking" the solution to behave like one outer branch on the left $y_l(x) = x + 1$, and like a different outer branch on the right $y_r(x) = x - 2$. A rapid transition inside $(0, 1)$ becomes unavoidable. That rapid transition is an *interior layer*.

Geometrically: away from the layer, the solution hugs one of the outer branches. Near one special point $x = x_0 \in (0, 1)$ it "jumps" from one branch to the other.

Location and structure of the interior layer

We introduce an unknown transition point x_0 where the rapid change occurs and where $y(x_0) = 0$. Near $x = x_0$, we expect y to vary sharply over a thin region of width $O(\varepsilon^\alpha)$ for some $\alpha > 0$.

Let

$$\bar{x} = \frac{x - x_0}{\varepsilon^\alpha}, \qquad y(x) = Y(\bar{x}).$$

Then

$$y' = \frac{1}{\varepsilon^\alpha} Y', \qquad y'' = \frac{1}{\varepsilon^{2\alpha}} Y''.$$

Substitute into the full equation

$$\varepsilon y'' = y\, y' - y.$$

We obtain

$$\varepsilon \cdot \frac{1}{\varepsilon^{2\alpha}} Y'' = Y \cdot \frac{1}{\varepsilon^\alpha} Y' - Y,$$

or

$$\varepsilon^{1-2\alpha} Y'' = \varepsilon^{-\alpha} Y Y' - Y.$$

To identify the correct thickness of the layer, we balance the leading terms. The two dominant terms near the jump must be the second derivative on the left and the nonlinear convective term YY' on the right. That balance requires

$$\varepsilon^{1-2\alpha} \sim \varepsilon^{-\alpha} \quad \Longrightarrow \quad 1 - 2\alpha = -\alpha \quad \Longrightarrow \quad \alpha = 1.$$

Thus the interior layer thickness is $O(\varepsilon)$:

$$\bar{x} = \frac{x - x_0}{\varepsilon}.$$

With $\alpha = 1$, the equation becomes

$$\varepsilon^{-1} Y'' = \varepsilon^{-1} Y Y' - Y.$$

Multiply both sides by ε:

$$Y'' = YY' - \varepsilon Y.$$

At the leading order, we obtain

$$Y_0'' = Y_0 Y_0'.$$

Integrate in $\bar{x}$:

$$Y_0' = \frac{1}{2} Y_0^2 + A.$$

This is separable:

$$\frac{dY}{\frac{1}{2}Y^2 + A} = d\bar{x},$$

which admits three types of solutions depending on whether A is negative, positive, or zero. The solutions are

$$Y_0 = B \frac{1 - De^{B\bar{x}}}{1 + De^{B\bar{x}}}, \quad Y_0 = B \tan(C - B\bar{x}/2), \quad Y_0 = \frac{2}{C - \bar{x}}.$$

Matching

The matching conditions are

$$Y_0(-\infty) = y_l(x_0) \quad \text{and} \quad Y_0(\infty) = y_r(x_0)$$

yielding the choice of type-one solution and

$$\begin{cases} B = x_0 + 1 \\ -B = x_0 - 2 \end{cases} \quad \text{or} \quad \begin{cases} B = \dfrac{3}{2} \\ x_0 = \dfrac{1}{2} \end{cases}$$

To determine the value of D we use $y(x_0) = 0$ which forces $D = 1$.

Composite solution

$$y \sim x + 1 + Y_0(\bar{x}) - \frac{3}{2} \quad \text{for } x \in [0, \frac{1}{2}] \quad y \sim x - 2 + Y_0(\bar{x}) + \frac{3}{2} \quad \text{for } x \in [\frac{1}{2}, 1].$$

or simply

$$y \sim x + Y_0(\bar{x}) - \frac{1}{2} \quad \text{for } x \in [0, 1].$$

4 Corner Layers (Holmes §2.6)

Consider

$$\varepsilon y'' + (x - \tfrac{1}{2})y' - y = 0, \qquad 0 < x < 1, \qquad y(0) = 2,\ y(1) = 3.$$

Unlike a standard boundary layer, here the singular behavior happens at an *interior point* $x = \frac{1}{2}$, where the coefficient of y' vanishes. The solution is continuous there, but its slope changes abruptly. This produces what Holmes calls a *corner layer*.

Outer problem (away from $x = \frac{1}{2}$)

Set $\varepsilon = 0$:

$$(x - \tfrac{1}{2})y_0' - y_0 = 0 \quad \Longrightarrow \quad \frac{y_0'}{y_0} = \frac{1}{x - \frac{1}{2}}.$$

Integrate:

$$\ln|y_0| = \ln|x - \tfrac{1}{2}| + C \quad \Longrightarrow \quad y_0(x) = K(x - \tfrac{1}{2}).$$

Important: this y_0 is only valid on each side of $x = \frac{1}{2}$ separately, because the ODE degenerates at $x = \frac{1}{2}$. So we solve with two different constants:

$$y_0^-(x) = A(x - \tfrac{1}{2}) \quad \text{for } 0 < x < \tfrac{1}{2}, \qquad y_0^+(x) = B(x - \tfrac{1}{2}) \quad \text{for } \tfrac{1}{2} < x < 1.$$

Use the boundary conditions:

$$y_0^-(0) = 2 \quad \Rightarrow \quad 2 = A(0 - \tfrac{1}{2}) = -\tfrac{1}{2}A \quad \Rightarrow \quad A = -4,$$

so

$$y_0^-(x) = -4(x - \tfrac{1}{2}) = -4x + 2.$$

Similarly,

$$y_0^+(1) = 3 \quad \Rightarrow \quad 3 = B(1 - \tfrac{1}{2}) = \tfrac{1}{2}B \quad \Rightarrow \quad B = 6,$$

so

$$y_0^+(x) = 6(x - \tfrac{1}{2}) = 6x - 3.$$

Observe:

$$y_0^-\left(\tfrac{1}{2}\right) = 0, \qquad y_0^+\left(\tfrac{1}{2}\right) = 0,$$

so the outer solution is *continuous* at $x = \frac{1}{2}$.
But the slopes disagree:

$$(y_0^-)'(x) = -4, \quad (y_0^+)'(x) = 6.$$

So the outer solution predicts a sharp *kink* (corner) at $x = \frac{1}{2}$: same value, different slope.

A corner layer appears to smooth that slope jump.

Corner-layer scaling

We zoom in near $x = \frac{1}{2}$. Let

$$\eta = \frac{x - \frac{1}{2}}{\delta}, \qquad y(x) = Y(\eta),$$

with δ to be determined. Then

$$y' = \frac{1}{\delta} Y', \qquad y'' = \frac{1}{\delta^2} Y'', \qquad x - \tfrac{1}{2} = \delta\eta.$$

Substitute into

$$\varepsilon y'' + (x - \tfrac{1}{2}) y' - y = 0 :$$

$$\varepsilon \frac{1}{\delta^2} Y'' + (\delta\eta) \frac{1}{\delta} Y' - Y = 0,$$

i.e.

$$\frac{\varepsilon}{\delta^2} Y'' + \eta Y' - Y = 0.$$

We now choose δ so that the two derivatives Y'' and $\eta Y'$ both appear at leading order. That requires

$$\frac{\varepsilon}{\delta^2} = O(1) \quad \Longrightarrow \quad \delta = \sqrt{\varepsilon}.$$

Therefore, the correct inner variable is

$$\eta = \frac{x - \frac{1}{2}}{\sqrt{\varepsilon}}, \qquad \text{(corner layer of thickness } O(\sqrt{\varepsilon})\text{)}.$$

With $\delta = \sqrt{\varepsilon}$, the rescaled equation becomes

$$Y'' + \eta Y' - Y = 0.$$

This is the *corner-layer equation*. We note that $Y = \eta$ is a solution to the second order linear ode above. We can use method of reduction of order to obtain another independent solution.

Matching

Left as homework.

Composite solution

Left as homework.