Matched Asymptotic Expansions

1 Introduction

Matched asymptotic expansions are used to construct approximations to solutions of singularly perturbed problems, where the small parameter ε multiplies the highest derivative.

A **singular perturbation problem** is one for which setting $\varepsilon = 0$ reduces the order of the differential equation, leading to loss of boundary conditions. The solution typically exhibits thin regions (boundary or interior layers) where the solution changes rapidly.

This method originated in fluid mechanics, particularly in Prandtl's (1905) analysis of the boundary layer in viscous flow past a solid surface.

Prandtl's Original Boundary-Layer Problem (1905)

The historical origin of matched asymptotic expansions traces back to Ludwig Prandtl's 1904–1905 analysis of viscous flow with very small viscosity. He studied the steady incompressible Navier–Stokes equations for flow past a flat plate or solid wall,

$$\begin{aligned} u\,u_x + v\,u_y &= -\frac{1}{\rho}p_x + \nu(u_{xx} + u_{yy}), \\ u\,v_x + v\,v_y &= -\frac{1}{\rho}p_y + \nu(v_{xx} + v_{yy}), \\ u_x + v_y &= 0, \end{aligned}$$

with boundary conditions

$$u = v = 0 \quad \text{at } y = 0, \qquad u \to U_\infty(x),\ v \to 0 \quad \text{as } y \to \infty.$$

When the viscosity ν tends to zero, the equations reduce to the inviscid Euler system, which cannot satisfy the no-slip condition. Prandtl observed that viscous effects remain important only in a *thin layer* adjacent to the wall. Introducing a stretched coordinate

$$Y = \frac{y}{\delta(x)}, \qquad \delta(x) \sim \sqrt{\frac{\nu x}{U_\infty}},$$

and neglecting small terms, he obtained the **boundary-layer equations**

$$\begin{aligned} u\,u_x + v\,u_y &= U_e\,U_{e,x} + \nu\,u_{yy}, \\ u_x + v_y &= 0, \end{aligned}$$

with

$$u(x,0) = v(x,0) = 0, \qquad u(x,y) \to U_e(x) \text{ as } y \to \infty.$$

For uniform external flow ($U_e(x) = U_\infty$) the equations simplify to

$$u\,u_x + v\,u_y = \nu\,u_{yy}, \qquad u_x + v_y = 0,$$

the problem Prandtl analyzed. In similarity variables

$$\eta = y\sqrt{\frac{U_\infty}{2\nu x}}, \qquad \psi(x,y) = \sqrt{2\nu U_\infty x}\, f(\eta),$$

it reduces to the celebrated **Blasius equation**

$$f''' + f\,f'' = 0, \qquad f(0) = f'(0) = 0,\ f'(\infty) = 1.$$

Prandtl's insight was that the full flow field can be described by *matching* the inviscid outer solution with the viscous inner (boundary-layer) solution. This is conceptually identical to the modern method of matched asymptotic expansions introduced later by Friedrichs, Lagerstrom, and Van Dyke.

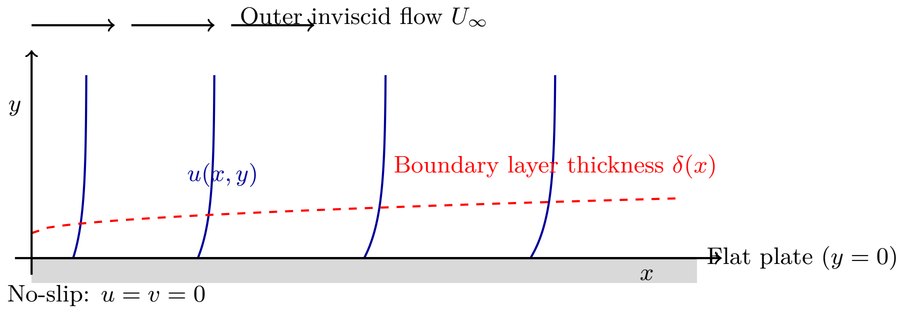

Figure 1: Schematic of Prandtl's boundary-layer problem. Viscosity effects are confined to a thin region near the wall where $u(x, 0) = v(x, 0) = 0$. Outside the layer, the flow is nearly inviscid with $u(x, y) \to U_\infty$. The boundary-layer thickness $\delta(x) \sim \sqrt{\nu x/U_\infty}$ grows slowly downstream.

2 Example Problem

We consider the boundary-value problem

$$\varepsilon y'' + 2y' + 2y = 0, \qquad 0 < x < 1, \tag{1}$$

with boundary conditions

$$y(0) = 0, \qquad y(1) = 1. \tag{2}$$

If $\varepsilon = 0$, equation (1) becomes first order, and so only one boundary condition can be imposed. This signals a **singular perturbation problem**, and we expect a thin boundary layer near one end of the domain.

2.1 Exact Solution of the Boundary Value Problem

To verify the asymptotic results later, we can compute the exact solution of (1)–(2).

The characteristic equation is

$$\varepsilon r^2 + 2r + 2 = 0, \qquad r = \frac{-2 \pm \sqrt{4 - 8\varepsilon}}{2\varepsilon} = \frac{-1 \pm \sqrt{1 - 2\varepsilon}}{\varepsilon}.$$

Denote

$$r_1 = \frac{-1 + \sqrt{1 - 2\varepsilon}}{\varepsilon}, \qquad r_2 = \frac{-1 - \sqrt{1 - 2\varepsilon}}{\varepsilon}.$$

The general solution is

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}.$$

From the boundary conditions,

$$\begin{cases} A + B = 0, \\ Ae^{r_1} + Be^{r_2} = 1, \end{cases} \qquad \Rightarrow \qquad A = \frac{1}{e^{r_1} - e^{r_2}}, \quad B = -A.$$

Hence

$$\boxed{y(x) = \frac{e^{r_1 x} - e^{r_2 x}}{e^{r_1} - e^{r_2}}.} \tag{3}$$

For small ε, note that

$$r_1 \approx -1 + \varepsilon, \qquad r_2 \approx -\frac{2}{\varepsilon} + 1,$$

so the term involving $e^{r_2 x}$ decays extremely rapidly near $x = 0$, creating a thin boundary layer there.

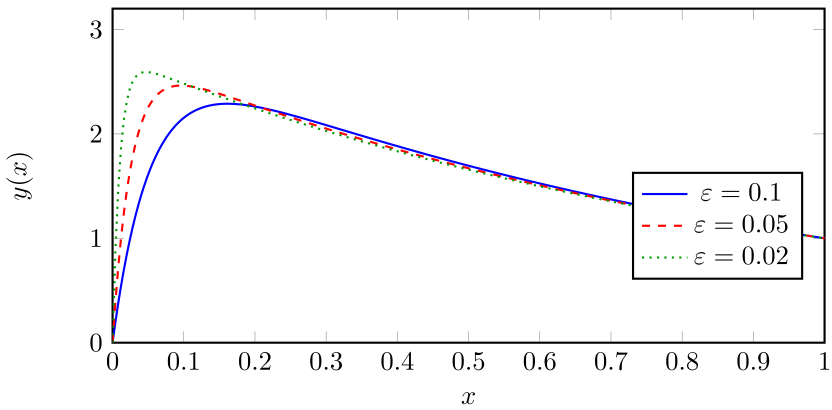

Figure 2: Exact solution (3) for several small values of ε. As $\varepsilon \to 0$, a sharp boundary layer develops near $x = 0$, with $y(0^+) \approx e$.

3 Step 1: Outer Solution

Assume an expansion in powers of ε:

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots .$$

Substitute into (1):

$$\varepsilon(y_0'' + \varepsilon y_1'' + \cdots) + 2(y_0' + \varepsilon y_1' + \cdots) + 2(y_0 + \varepsilon y_1 + \cdots) = 0.$$

The $O(1)$ equation gives

$$y_0' + y_0 = 0 \quad \Rightarrow \quad y_0 = Ae^{-x}.$$

Only one constant A appears, so y_0 cannot satisfy both conditions (2). Hence, the outer solution is valid away from the boundary layer.

We will assume the boundary layer is near $x = 0$, and require $y_0(1) = 1$, giving

$$y_0(x) = e^{1-x}. \tag{4}$$

4 Step 2: Inner (Boundary-Layer) Solution

Introduce a stretched variable near $x = 0$:

$$\bar{x} = \frac{x}{\varepsilon^\alpha}, \qquad \alpha > 0,$$

and let $Y(\bar{x}) = y(x)$. By the chain rule,

$$\frac{d}{dx} = \frac{1}{\varepsilon^\alpha}\frac{d}{d\bar{x}}, \qquad \frac{d^2}{dx^2} = \frac{1}{\varepsilon^{2\alpha}}\frac{d^2}{d\bar{x}^2}.$$

Equation (1) becomes

$$\varepsilon^{1-2\alpha}Y'' + 2\varepsilon^{-\alpha}Y' + 2Y = 0.$$

Balancing the dominant terms gives

$$1 - 2\alpha = -\alpha \quad \Rightarrow \quad \alpha = 1.$$

Hence the boundary layer has width $O(\varepsilon)$.

With $\bar{x} = x/\varepsilon$, the leading-order equation is

$$Y_0'' + 2Y_0' = 0, \qquad Y_0(0) = 0,$$

whose solution is

$$Y_0(\bar{x}) = A(1 - e^{-2\bar{x}}). \tag{5}$$

5 Step 3: Matching

Since the inner and outer approximations describe the same function, they must agree in the overlap region. Thus,

$$\lim_{\bar{x}\to\infty} Y_0(\bar{x}) = \lim_{x\to 0^+} y_0(x).$$

From (5) and (4),

$$A = e^1.$$

Hence,

$$Y_0(\bar{x}) = e^1(1 - e^{-2\bar{x}}).$$

6 Step 4: Composite Expansion

We can form a uniform approximation valid for $0 \le x \le 1$ by adding the two approximations and subtracting their common part:

$$y_{\text{comp}}(x) = y_0(x) + Y_0(x/\varepsilon) - y_0(0).$$

Therefore,

$$y_{\text{comp}}(x) = e^{1-x} - e^{1-2x/\varepsilon}. \tag{6}$$

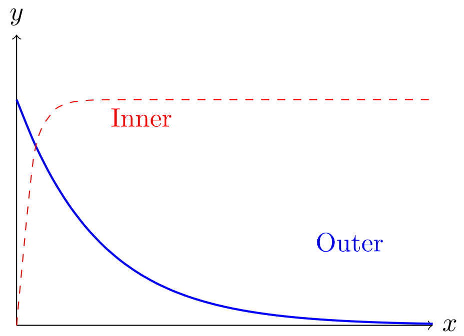

Figure: Outer and inner approximations with overlap region.

7 Step 5: Matching Revisited

In the first matching step we used the simple condition

$$Y_0(\infty) = y_0(0^+),$$

which works well for this problem. However, in more complicated singular-perturbation problems, either of these limits may fail to exist or become unbounded. To handle such cases, a more systematic procedure is needed.

Kaplun's Matching Principle

Following Kaplun and Lagerstrom (1988), we introduce an *intermediate variable*

$$x_\eta = \frac{x}{\eta(\varepsilon)}, \qquad \text{where } \varepsilon \ll \eta(\varepsilon) \ll 1.$$

This variable lies in the overlap region between the outer coordinate ($x = O(1)$) and the inner coordinate ($\bar{x} = x/\varepsilon = O(1)$).

The matching principle is then stated as follows:

(i) Express the **outer expansion** y_{outer} in terms of x_η and determine its domain of validity for $\eta_1(\varepsilon) \le \eta(\varepsilon) \le 1$.

(ii) Express the **inner expansion** y_{inner} in terms of x_η and determine its validity for $\varepsilon \leq \eta(\varepsilon) \leq \eta_2(\varepsilon)$.

(iii) If the two domains overlap ($\eta_1 < \eta_2$), require that the first terms of the two expansions be asymptotically equal in the overlap region.

This matching need not depend on any particular choice of $\eta(\varepsilon)$. The existence of such an overlap domain is known as **Kaplun's hypothesis on the domain of validity**. It cannot be proved in full generality but holds in all classical examples.

Example (Revisited)

For our problem, let $x_\eta = x/\varepsilon^\beta$ with $0 < \beta < 1$. Then, as shown by Holmes,

$$y_{\text{inner}} \sim A(1 - e^{-2x_\eta/\varepsilon^{1-\beta}}) + \cdots \sim A + \cdots, \qquad y_{\text{outer}} \sim e^{1-x_\eta\varepsilon^\beta} + \cdots \sim e^1 + \cdots.$$

Equating leading terms gives $A = e^1$, in agreement with our previous simple matching condition.

Thus, the intermediate-variable approach provides a general framework for matching inner and outer expansions systematically, and it forms the basis for more advanced asymptotic analysis.

8 Comparison with the Exact Solution

The exact solution (3) can be expanded for small ε to confirm that it agrees with (6) up to exponentially small terms $O(e^{-2/\varepsilon})$. Numerically, the composite approximation is already indistinguishable from the exact solution for $\varepsilon \lesssim 0.1$.

9 Key Concepts

1. **Outer region:** smooth behavior, expansion in ε with x fixed.
2. **Inner region:** rapid variation; introduce stretched variable $\bar{x} = x/\varepsilon$.
3. **Matching:** ensure asymptotic agreement in the overlap region.
4. **Composite expansion:** sum minus overlap for a uniform approximation.
5. **Validation:** compare to exact solution to assess accuracy.