Asymptotic Expansions and Rigorous Justifications

1 A Motivating Example:

Consider the singularly perturbed algebraic equation

$$\varepsilon x^2 + 2x - 1 = 0, \qquad \varepsilon \text{ small.} \tag{1}$$

Solving explicitly, we obtain

$$x_\pm(\varepsilon) = \frac{-1 \pm \sqrt{1+\varepsilon}}{\varepsilon}.$$

Expanding for small ε, we find

$$\sqrt{1+\varepsilon} = 1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + O(|\varepsilon|^3), \quad \text{as } \varepsilon \to 0.$$

Hence, there hold

$$x_+(\varepsilon) = \frac{1}{2} - \frac{\varepsilon}{8} + O(\varepsilon^2) \quad \text{and} \quad x_-(\varepsilon) = -\frac{2}{\varepsilon} - \frac{1}{2} + O(\varepsilon), \quad \text{as } \varepsilon \to 0.$$

As $\varepsilon \to 0$, the first root $x_+(\varepsilon)$ remains bounded while the second one $x_-(\varepsilon)$ blows up.

Rigorous Justifications

Regular root. Our goal is to establish:

Proposition 1. *There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $|\varepsilon| \le \varepsilon_0$, the equation admits a unique bounded root $x_+(\varepsilon)$ satisfying*

$$|x_+(\varepsilon) - \tfrac{1}{2} + \tfrac{\varepsilon}{8}| \le C\varepsilon^2,$$

so that

$$x_+(\varepsilon) = \frac{1}{2} - \frac{\varepsilon}{8} + O(\varepsilon^2).$$

Method 1: Applying Taylor's theorem. The above expansion is obtained formally by Taylor expansion. To make it rigorous, use the Lagrange form of the remainder in Taylor's theorem:

$$\sqrt{1+\varepsilon} = 1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + R_3(\varepsilon), \qquad R_3(\varepsilon) = \frac{f'''(\xi)}{3!}\,\varepsilon^3 \text{ for some } \xi = \xi(\varepsilon) \text{ lying between 0 and } \varepsilon,$$

with $f(x) = \sqrt{1+x}$. For $\varepsilon_0 < 1$ fixed, setting $C := \sup_{x\in[-\varepsilon_0,\varepsilon_0]} \frac{1}{6}|f'''(x)|$, we obtain $|R_3(\varepsilon)| \le C\varepsilon^3$. Thus, for $|\varepsilon| \le \varepsilon_0$, there holds

$$x_+(\varepsilon) = \frac{1}{2} - \frac{\varepsilon}{8} + \frac{R_3(\varepsilon)}{\varepsilon}, \quad |x_+(\varepsilon) - \tfrac{1}{2} + \tfrac{\varepsilon}{8}| \le C\varepsilon^2.$$

Method 2: Direct estimates. We can estimate directly, without Taylor's theorem, that for fixed $\varepsilon_0 < 1$ and some $C = C(\varepsilon_0) > 0$, there holds

$$\left| x_+(\varepsilon) - \frac{1}{2} + \frac{\varepsilon}{8} \right| \le C\varepsilon^2, \quad |\varepsilon| \le \varepsilon_0.$$

Sketch of the estimation. Using $s = \sqrt{1+\varepsilon}$ so that $\varepsilon = s^2 - 1$, we compute

$$x_+ - \frac{1}{2} + \frac{\varepsilon}{8} = \frac{-1+\sqrt{1+\varepsilon}}{\varepsilon} - \frac{1}{2} + \frac{\varepsilon}{8} = \varepsilon^2 \frac{s+3}{8(s+1)^3},$$

so $|x_+ - \frac{1}{2} + \frac{\varepsilon}{8}| \le C\varepsilon^2$ with $C = 1$.

Remark 1. *A drawback of methods 1 and 2 are that they rely on the closed form of roots derived from the quadratic formula.*

Method 3: Applying implicit function theorem (IFT).

Considering the function $f(x,\varepsilon) = \varepsilon x^2 + 2x - 1$ and applying analytic IFT at $(\frac{1}{2}, 0)$, there exists $\varepsilon_0 > 0$ and a unique analytic function $x(\varepsilon)$ for $|\varepsilon| < \varepsilon_0$ with $x(0) = 1/2$ and $f(x(\varepsilon), \varepsilon) \equiv 0$. Analyticity means $x(\varepsilon)$ has a convergent power series

$$x(\varepsilon) = \frac{1}{2} + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

valid for $|\varepsilon| < \varepsilon_0$. Matching coefficients recovers $a_1 = -1/8$ etc., so the formal expansion is not just formal — it converges to the true root. Now

$$|x(\varepsilon) - \frac{1}{2} + \frac{\varepsilon}{8}| = |a_2\varepsilon^2 + \cdots| = \varepsilon^2|a_2 + \cdots| \leq C\varepsilon^2,$$

where $C = \max_{|\varepsilon| \geq \frac{1}{2}\varepsilon_0} |a_2 + \cdots|$.

Remark 2. *The IFT gives existence, uniqueness (no other root near (1/2)), and convergence of the Taylor series.*

Method 4: Constructing contraction mappings.

Rearrange the equation (1) as a fixed-point problem

$$x = F(x;\varepsilon) := \frac{1 - \varepsilon x^2}{2}.$$

We seek a fixed point near $x_0 = \frac{1}{2}$. Let $\varepsilon_0 > 0$ be small and consider the closed ball

$$B := \Big\{x \in \mathbb{R} : |x - \tfrac{1}{2}| \leq r\Big\},$$

where r will be chosen proportional to ε_0.

Step 1: F maps B into itself. For $x \in B$ we have

$$|F(x;\varepsilon) - \tfrac{1}{2}| = \Big|\frac{1 - \varepsilon x^2}{2} - \frac{1}{2}\Big| = \frac{|\varepsilon|}{2}|x|^2 \leq \frac{|\varepsilon|}{2}\Big(\frac{1}{2} + r\Big)^2.$$

Hence if we require

$$r \geq \frac{|\varepsilon|}{2}\Big(\frac{1}{2} + r\Big)^2,$$

then F maps B into itself. For instance, with $r = \varepsilon_0$ this holds for all $|\varepsilon| \leq \varepsilon_0$ provided $\varepsilon_0 \leq \frac{1}{2}$.

Step 2: F is a contraction on B. We compute

$$\partial_x F(x;\varepsilon) = -\varepsilon x.$$

Thus for $x, y \in B$,

$$|F(x;\varepsilon) - F(y;\varepsilon)| \leq |\varepsilon|\,|z(x,y;\varepsilon)|\,|x - y| \leq |\varepsilon|\Big(\frac{1}{2} + \varepsilon_0\Big)|x - y|.$$

Since $\varepsilon_0 \leq \frac{1}{2}$, for $|\varepsilon| \leq \varepsilon_0$, the Lipschitz constant $\theta = \theta(\varepsilon, \varepsilon_0) := |\varepsilon|\Big(\frac{1}{2} + \varepsilon_0\Big)$ is at most $|\varepsilon|(< 1)$, so F is a contraction on B.

Combining **Step 1** and **Step 2**, we conclude that for a fixed $\varepsilon_0 \leq \frac{1}{2}$, for $|\varepsilon| \leq \varepsilon_0$, the map

$$F(\,\cdot\,;\varepsilon) : x \mapsto \frac{1 - \varepsilon x^2}{2}$$

is a contraction mapping on $B = \{x \in \mathbb{R} : |x - \frac{1}{2}| \leq \varepsilon_0\}$ satisfying

$$|F(x;\varepsilon) - F(y;\varepsilon)| \leq \theta(\varepsilon;\varepsilon_0)|x - y|, \quad \text{for } x, y \in B.$$

Step 3: existence, uniqueness, and estimate. By Banach's fixed point theorem, there exists a unique fixed point $x_+(\varepsilon) \in B$ such that $x_+(\varepsilon) = F(x_+(\varepsilon);\varepsilon)$. Now,

$$|x_+(\varepsilon) - \frac{1}{2}| = \Big|F(x_+(\varepsilon);\varepsilon) - F\left(\tfrac{1}{2};\varepsilon\right) + F\left(\tfrac{1}{2};\varepsilon\right) - \frac{1}{2}\Big|$$

$$\leq \Big|F(x_+(\varepsilon);\varepsilon) - F\left(\tfrac{1}{2};\varepsilon\right)\Big| + \Big|F\left(\tfrac{1}{2};\varepsilon\right) - \frac{1}{2}\Big|$$

$$\leq \theta|x_+(\varepsilon) - \frac{1}{2}| + \frac{|\varepsilon|}{8}$$

for some contraction constant $\theta < 1$. So the fixed point satisfies

$$|x(\varepsilon) - \tfrac{1}{2}| \leq \frac{1}{1-\theta}\frac{|\varepsilon|}{8},$$

whence $x(\varepsilon) = \frac{1}{2} + \mathcal{O}(|\varepsilon|)$.
To show

$$|x_+(\varepsilon) - \tfrac{1}{2} + \tfrac{\varepsilon}{8}| \leq C\varepsilon^2,$$

we can similarly estimate

$$\begin{aligned}|x_+(\varepsilon) - \frac{1}{2} + \frac{\varepsilon}{8}| &= \left|F(x_+(\varepsilon);\varepsilon) - F\left(\tfrac{1}{2} - \tfrac{\varepsilon}{8};\varepsilon\right)\right| + \left|F\left(\tfrac{1}{2} - \tfrac{\varepsilon}{8};\varepsilon\right) - (\frac{1}{2} - \frac{\varepsilon}{8})\right| \\ \leq \theta|x_+(\varepsilon) - \frac{1}{2} + \frac{\varepsilon}{8}| + \frac{8-\varepsilon}{128}\varepsilon^2,\end{aligned}$$

provided that $\frac{1}{2} - \frac{\varepsilon}{8} \in B$ which certainly holds. The latter inequality yields that

$$|x_+(\varepsilon) - \frac{1}{2} + \frac{\varepsilon}{8}| \leq \frac{8-\varepsilon}{128(1-\theta)}\varepsilon^2.$$

In general, for $|\varepsilon| \leq \varepsilon_0$ set the iteration sequence $\{y_n(\varepsilon)\}_{n=0}^{\infty}$ by $y_0 = x_0 = \frac{1}{2}$, $y_n(\varepsilon) = F(y_{n-1}(\varepsilon);\varepsilon)$ for $n \geq 1$. We justify that $y_n(\varepsilon) \in B$ for $n \geq 0$ inductively. The base case $y_0 \in B$ holds. By the invariance of $F(\,\cdot\,;\varepsilon)$ on B, given that $y_{n-1}(\varepsilon) \in B$, we have $y_n(\varepsilon) = F(y_{n-1}(\varepsilon);\varepsilon) \in B$. This enables us to estimate

$$\begin{aligned}|x_+(\varepsilon) - y_n(\varepsilon)| &= |F(x_+(\varepsilon);\varepsilon) - F\left(y_{n-1}(\varepsilon);\varepsilon\right)| \\ &\leq |\varepsilon|(\frac{1}{2} + \varepsilon_0)|x_+(\varepsilon) - x_{n-1}(\varepsilon)| \\ &\leq \left(|\varepsilon|(\frac{1}{2} + \varepsilon_0)\right)^n |x_+(\varepsilon) - x_0(\varepsilon)| \leq \left(|\varepsilon|(\frac{1}{2} + \varepsilon_0)\right)^n \frac{1}{1-\theta}\frac{|\epsilon|}{8} \sim \mathcal{O}(|\varepsilon|^{n+1}).\end{aligned}$$

Note that $y_n(\varepsilon)$ is a polynomial of ε. Warning! If $x_n(\varepsilon)$ is the unique polynomial of ε of degree n such that

$$x_+(\varepsilon) = x_n(\varepsilon) + \mathcal{O}(|\varepsilon|^{n+1}).$$

we do not have $x_n(\varepsilon) = y_n(\varepsilon)$ for $n \geq 2$ due to the nonlinearity of iteration map $f(\cdot, \varepsilon)$. But $x_n(\varepsilon) = y_n(\varepsilon) + \mathcal{O}(|\varepsilon|^{n+1})$.
Homework: design an algorithm computing $x_n(\varepsilon)$ inductively.

Singular root. Our goal is to justify the formal Laurent expansion

$$x_-(\varepsilon) \sim -\frac{2}{\varepsilon} - \frac{1}{2} + O(|\varepsilon|)$$

by establishing:

Proposition 2. *There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $|\varepsilon| \leq \varepsilon_0$, the equation admits a unique unbounded root $x_-(\varepsilon)$ satisfying*

$$|x_-(\varepsilon) + \frac{2}{\varepsilon} + \frac{1}{2}| \leq C|\varepsilon|,$$

so that

$$x_-(\varepsilon) = -\frac{2}{\varepsilon} - \frac{1}{2} + O(|\varepsilon|).$$

desingularization by change of variables. Set

$$X := \varepsilon x,$$

so the equation $\varepsilon x^2 + 2x - 1 = 0$ becomes

$$X^2 + 2X - \varepsilon = 0.$$

The large root corresponds to X near -2 (since $x \sim -2/\varepsilon$). Write

$$X = -2 + y,$$

so y is expected to be small (indeed $y = O(\varepsilon)$). The equation for y is

$$(-2+y)^2 + 2(-2+y) - \varepsilon = 0 \quad \Longleftrightarrow \quad y^2 - 2y = \varepsilon,$$

or equivalently

$$y = S(y) := \frac{y^2 - \varepsilon}{2}.$$

Then applying any method you like to continue the justification.

2 Nonlinear ODE Example:

Consider the initial value problem

$$x''(t) = -\frac{1}{(1+\varepsilon x(t))^2}, \qquad x(0) = 0, \quad x'(0) = 1, \tag{2}$$

where ε is small.

Formally, we expect an asymptotic expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2), \tag{3}$$

where the big O is understood in some functional space. We now compute $x_0(t)$ and $x_1(t)$. The RHS of (2) expands as

$$-\frac{1}{(1+\varepsilon x)^2} = -1 + 2\varepsilon x + \mathcal{O}(\varepsilon^2).$$

Inserting the ansatz (3) into (2), we obtain

$$x_0'' = -1, \qquad x_0(0) = 0, \quad x_0'(0) = 1 \quad \Rightarrow \quad x_0(t) = t - \tfrac{1}{2}t^2.$$

and

$$x_1''(t) = 2x_0(t), \qquad x_1(0) = x_1'(0) = 0,$$

so x_1 is obtained by two integrations:

$$x_1(t) = \int_0^t \int_0^\tau 2x_0(s)dsd\tau = \int_0^t (t-s)2x_0(s)ds.$$

We now show that the asymptotic expansion (3) can *rigorously justified* by a contraction mapping argument and our goal is to prove the following theorem.

Theorem 1 (rigorous first-order expansion)**.** *Fix $T > 0$. There exists $\varepsilon_0 = \varepsilon_0(T) > 0$ and $C = C(T) > 0$ such that for all $0 \le \varepsilon \le \varepsilon_0$ the IVP*

$$x'' = -\frac{1}{(1+\varepsilon x)^2}, \quad x(0) = 0, \quad x'(0) = 1,$$

has a unique solution on $[0,T]$ and

$$x(t) = x_0(t) + \varepsilon x_1(t) + r(t), \qquad |r|_{C([0,T])} \le C\varepsilon^2,$$

where $x_0(t) = t - \frac{1}{2}t^2$ and $x_1(t) = \int_0^t (t-s)2x_0(s)ds$.

Define the remainder

$$r(t) := x(t) - x_0(t) - \varepsilon x_1(t).$$

We will show $|r| = O(\varepsilon^2)$ in a suitable norm by contraction mapping.

Step 1. Integral formulation

Integrating twice, we obtain

$$x(t) = t - \int_0^t (t-s)\frac{1}{(1+\varepsilon x(s))^2}\, ds.$$

Define the operator

$$(\mathcal{T}_\varepsilon x)(t) := t - \int_0^t (t-s)\frac{1}{(1+\varepsilon x(s))^2}\, ds.$$

A fixed point $x = \mathcal{T}_\varepsilon x$ solves (2).

Step 2. Functional setting.

Let

$$X_T = \{x \in C([0,T]) : x(0) = 0\}, \qquad \|x\| := \sup_{t\in[0,T]} |x(t)|.$$

For small enough $T > 0$, define the closed ball

$$B_M := \{x(\cdot) \in X_T : \|x(\cdot) - x_0(\cdot)\| \le M\}.$$

Step 3. Invariance.

Let $\varepsilon_0 > 0$ be fixed. For $|\varepsilon| \le \varepsilon_0$, we would like to decide how small ε_0 need to be in order for $\mathcal{T}_\varepsilon$ to be invariant on B_M. To this end, we compute

$$\begin{aligned}(\mathcal{T}_\varepsilon x)(t) - x_0(t) &= t - \int_0^t (t-s)\frac{1}{(1+\varepsilon x(s))^2}\, ds - (t - \frac{1}{2}t^2) \\ &= -\int_0^t (t-s)\left(\frac{1}{(1+\varepsilon x(s))^2} - 1\right) ds\end{aligned}$$

Set $M_1 := \|x_0(\cdot)\|$. We have for $x(\cdot) \in B_M$, $\|x(\cdot)\| \le M + M_1$. Let us require that

$$\varepsilon_0 < \frac{1}{M + M_1} \tag{4}$$

so that $(1+\varepsilon x(s))^{-1}$ makes sense. Setting $F(x) = (1+x)^{-2}$, we find that

$$\frac{1}{(1+\varepsilon x(s))^2} - 1 = F(\varepsilon x(s)) - F(0) = F'(\theta(\varepsilon x(s)))\varepsilon x(s) = \frac{-2\varepsilon x(s)}{\big(1+\theta(\varepsilon x(s))\big)^3}.$$

There then holds for $t \in [0,T]$

$$\begin{aligned}|(\mathcal{T}_\varepsilon x)(t) - x_0(t)| &\le \int_0^t (t-s)\left|\frac{1}{(1+\varepsilon x(s))^2} - 1\right| ds = \int_0^t (t-s)\left|\frac{-2\varepsilon x(s)}{\big(1+\theta(\varepsilon x(s))\big)^3}\right| ds \\ &\le \int_0^t (t-s)\frac{2(M+M_1)\varepsilon_0}{(1-\varepsilon_0(M+M_1))^3} ds = \frac{t^2(M+M_1)\varepsilon_0}{(1-\varepsilon_0(M+M_1))^3} \le \frac{T^2(M+M_1)\varepsilon_0}{(1-\varepsilon_0(M+M_1))^3}.\end{aligned}$$

We then choose ε_0 small enough so that

$$\frac{T^2(M+M_1)\varepsilon_0}{(1-\varepsilon_0(M+M_1))^3} \le M, \tag{5}$$

which guarantees that

$$\|(\mathcal{T}_\varepsilon x)(\cdot) - x_0(\cdot)\| \le M.$$

Step 4. Contraction estimate

Let $x_1, x_2 \in B_M$. Then

$$\begin{aligned}(\mathcal{T}_\varepsilon x_1)(t) - (\mathcal{T}_\varepsilon x_2)(t) &= \int_0^t (t-s)\left(\frac{1}{(1+\varepsilon x_1(s))^2} - \frac{1}{(1+\varepsilon x_2(s))^2}\right) ds \\ &= \int_0^t (t-s)\left(F(\varepsilon x_1(s)) - F(\varepsilon x_2(s))\right) ds \\ &= \int_0^t (t-s)F'(\theta(\varepsilon x_1(s), \varepsilon x_2(s))\varepsilon(x_1(s) - x_2(s)) ds\end{aligned}$$

Note that

$$\|\theta(\varepsilon x_1(s), \varepsilon x_2(s)\| \leq 2\varepsilon_0(M + M_1).$$

So we further require that

$$2\varepsilon_0(M + M_1) < 1 \tag{6}$$

to ensure the uniform boundedness of $F'(\theta(\varepsilon x_1(s), \varepsilon x_2(s))$. It follows that for $t \in [0, T]$

$$\begin{aligned} |(\mathcal{T}_\varepsilon x_1)(t) - (\mathcal{T}_\varepsilon x_2)(t)| &\leq \int_0^t (t-s) \frac{2}{(1 - 2\varepsilon_0(M + M_1))^3} \varepsilon \|x_1 - x_2\| ds \\ &\leq \frac{T^2 \varepsilon}{(1 - 2\varepsilon_0(M + M_1))^3} \|x_1 - x_2\|, \end{aligned}$$

whence

$$\|(\mathcal{T}_\varepsilon x_1)(\cdot) - (\mathcal{T}_\varepsilon x_2)(\cdot)\| \leq \frac{T^2 \varepsilon}{(1 - 2\varepsilon_0(M + M_1))^3} \|x_1 - x_2\| \leq \frac{T^2 \varepsilon_0}{(1 - 2\varepsilon_0(M + M_1))^3} \|x_1 - x_2\|.$$

$\mathcal{T}_\varepsilon$ is a contraction on B_M provided that

$$\frac{T^2 \varepsilon_0}{(1 - 2\varepsilon_0(M + M_1))^3} < 1. \tag{7}$$

By Banach's fixed point theorem, there exists a unique fixed point $x_\varepsilon(\cdot) \in X_T$ solving $x = \mathcal{T}_\varepsilon x$.
Homework: work out the rest of the proof and more.