$$e^x = 1 + x + \frac{x^2}{2} + O\left(x^3\right) \quad as \quad x \downarrow 0$$

$$e^x = 1 + x + \frac{x^2}{2} + o\left(x^2\right) \quad as \quad x \downarrow 0$$

Two criteria

Theorem 1.3.

1. If

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = L, \tag{1.7}$$

where $-\infty < L < \infty$, *then* $f = O(\phi)$ *as* $\varepsilon \downarrow \varepsilon_0$.

2. If

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = 0, \tag{1.8}$$

then $f = o(\phi)$ *as* $\varepsilon \downarrow \varepsilon_0$.

These are merely sufficient conditions as we will see next in a few examples.

1. Show the “exponential” example by applying the theorem. Need L’hospital rule.

2. Show that $\quad \epsilon \sin(\frac{1}{\epsilon}) = O(\epsilon) \quad \text{as} \quad \epsilon \downarrow 0$

3. Transcendentally (exponentially) small term. Show that

$$\text{for an arbitrary } \alpha > 0, \quad e^{-1/\epsilon} = o(\epsilon^\alpha) \quad \text{as} \quad \epsilon \downarrow 0$$

We say in this case that f is transcendentally small with respect to the power functions epsilon^alpha

Some properties

(a) $f = O(1)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow f$ is bounded as $\varepsilon \downarrow \varepsilon_0$.

(b) $f = o(1)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow f \to 0$ as $\varepsilon \downarrow \varepsilon_0$.

(c) $f = o(\phi)$ as $\varepsilon \downarrow \varepsilon_0 \Rightarrow f = O(\phi)$ as $\varepsilon \downarrow \varepsilon_0$ (but not vice versa).

1.4 Asymptotic approximations

Consider approximating $\quad f(\varepsilon) = \varepsilon^2 + \varepsilon^5$

by

$$g_1(\varepsilon) = \varepsilon^2, \quad g_2(\epsilon) = \frac{2}{3}\epsilon^2$$

The latter approximation is called a “lousy approximation” for its error is of the same order as the function we are using to approximate $f(\varepsilon)$

The example gives rise to

Definition 1.2. Given $f(\varepsilon)$ and $\phi(\varepsilon)$, we say that $\phi(\varepsilon)$ is an *asymptotic approximation* to $f(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$ whenever $f = \phi + o(\phi)$ as $\varepsilon \downarrow \varepsilon_0$. In this case we write $f \sim \phi$ as $\varepsilon \downarrow \varepsilon_0$.

1. The definition ensures that the error function is of higher order than the approximating function.
2. Given that phi is non-zero near epsilon_0, then

$$\lim_{\varepsilon \downarrow 0} \frac{f(\epsilon)}{\phi(\epsilon)} = 1 \Rightarrow f \sim \phi$$

Demo: Examples 1、2 on page 8, Read example 3 on page 9.

1.4.1 Asymptotic Expansions

Example 1 on page 8 shows that an asymptotic approximation is not unique, also it does not say much about the accuracy.
A cure is asymptotic expansion as defined next.

Definition 1.3.

1. The functions $\phi_1(\varepsilon)$, $\phi_2(\varepsilon)$, $\ldots$ form an *asymptotic sequence*, or are *well ordered*, as $\varepsilon \downarrow \varepsilon_0$ if and only if $\phi_{m+1} = o(\phi_m)$ as $\varepsilon \downarrow \varepsilon_0$ for all m.
2. If $\phi_1(\varepsilon)$, $\phi_2(\varepsilon)$, $\ldots$ is an asymptotic sequence, then $f(\varepsilon)$ has an *asymptotic expansion* to n terms, with respect to this sequence, if and only if

$$f = \sum_{k=1}^{m} a_k \phi_k + o(\phi_m) \text{ for } m = 1, 2, \ldots, n \quad \text{as } \varepsilon \downarrow \varepsilon_0, \tag{1.10}$$

where the a_k are independent of ε. In this case we write

$$f \sim a_1\phi_1(\varepsilon) + a_2\phi_2(\varepsilon) + \cdots + a_n\phi_n(\varepsilon) \quad \text{as } \varepsilon \downarrow \varepsilon_0. \tag{1.11}$$

The ϕ_k are called the scale or gauge or basis functions.

What scale functions are often used?

Generalized power series functions

1. $\phi_1 = (\varepsilon - \varepsilon_0)^\alpha$, $\phi_2 = (\varepsilon - \varepsilon_0)^\beta$, $\phi_3 = (\varepsilon - \varepsilon_0)^\gamma$, $\ldots$, where $\alpha < \beta < \gamma < \cdots$.

Exponentially small sequence

2. $\phi_1 = 1$, $\phi_2 = \mathrm{e}^{-1/\varepsilon}$, $\phi_3 = \mathrm{e}^{-2/\varepsilon}$, $\ldots$.

How do we find an asymptotic expansion of a certain function?

Demo: Example 2 and 3 on pages 11.

Demo: Example on page 12. transcendentally small term not shown in the asymptotic expansion.

reading 1.4.2

1.4.1 Manipulating Asymptotic Expansions

Consider two asymptotic expansions

$$f(x,\epsilon) \sim a_1(x)\phi_1(\epsilon) + a_2(x)\phi_2(\epsilon) + \cdots + a_n(x)\phi_n(\epsilon),$$
$$g(x,\epsilon) \sim b_1(x)\phi_1(\epsilon) + b_2(x)\phi_2(\epsilon) + \cdots + b_n(x)\phi_n(\epsilon),$$

with respect to the same basis functions.
What can we say about the expansions of

$$f(x,\epsilon) \pm g(x,\epsilon), \quad f(x,\epsilon)g(x,\epsilon), \quad \frac{d}{dx}f(x,\epsilon), \quad \int f(x,\epsilon)$$

Addition and subtraction

$$f(x,\epsilon) + g(x,\epsilon) \sim (a_1(x)+b_1(x))\phi_1(\epsilon) + (a_2(x)+b_2(x))\phi_2(\epsilon) + \cdots + (a_n(x)+b_n(x))\phi_n(\epsilon)$$

Multiplication

See Exercise 1.12

Differentiation

given that

$$f(x,\epsilon) \sim a_1(x)\phi_1(\epsilon) + a_2(x)\phi_2(\epsilon) + \cdots + a_n(x)\phi_n(\epsilon),$$

it is in general NOT true that

$$f'(x,\epsilon) \sim a_1'(x)\phi_1(\epsilon) + a_2'(x)\phi_2(\epsilon) + \cdots + a_n'(x)\phi_n(\epsilon).$$

Example 1

$$f(x,\epsilon) = e^{-x/\epsilon}\sin(e^{x/\epsilon}).$$

consider the basis functions $\{1, \epsilon, \epsilon^2, \cdots\}$.

then $f(x,\epsilon) \sim 0\cdot\epsilon + 0\cdot\epsilon^2 + 0\cdot\epsilon^3 + \cdots$

But we compute to find that

$$\frac{d}{dx}f(x,\epsilon) = -\frac{1}{\epsilon}e^{-x/\epsilon}\sin(e^{x/\epsilon}) + \frac{1}{\epsilon}\cos(e^{x/\epsilon}).$$

where the foregoing term is still exponentially small but the latter term does not have an asymptotic expansion w.r.t. the basis functions.

Integration

given that

$$f(x,\epsilon) \sim a_1(x)\phi_1(\epsilon) + a_2(x)\phi_2(\epsilon) + \cdots + a_n(x)\phi_n(\epsilon),$$

with a_1(x), a_2(x),…,a_n(x) being integrable then we have

$$\int_a^b f(x,\epsilon)dx \sim \int_a^b a_1(x)dx\phi_1(\epsilon) + \int_a^b a_2(x)dx\phi_2(\epsilon) + \cdots + \int_a^b a_n(x)dx\phi_n(\epsilon),$$

Example 1

find the asymptotic expansion of

$$f(\epsilon) = \int_0^1 e^{\epsilon x^2}dx,$$

we have

$$e^{\epsilon x^2} = \sum_{n=0}^{N}\frac{(\epsilon x^2)^n}{n!} + \mathcal{O}(\epsilon^{N+1}), \quad \text{uniformly for} \quad \text{x} \in [0,1].$$

$$\int_0^1 e^{\epsilon x^2}dx = \sum_{n=0}^{N}\int_0^1 \frac{(\epsilon x^2)^n}{n!}dx + \mathcal{O}(\epsilon^{N+1}).$$

Also Demo example 2 on bottom of page 16.

1.5 Asymptotic Solution of Algebraic and Transcendental Equations

Example 1 (a regular perturbation problem)

$$x^2 + 2\epsilon x - 1 = 0$$

Exact solutions are

$$x_\pm = \frac{-2\epsilon \pm \sqrt{4+4\epsilon^2}}{2} = -\epsilon \pm \sqrt{1+\epsilon^2}$$

which is analytic at epsilon=0.

Now, suppose one does not know quadratic formula but know some theory of asymptotic analysis. He would try to expand the series as

$$x = x_0 + x_1\epsilon^\alpha + o(\epsilon^\alpha)$$

work out the computation and determine x_0, x_1, alpha

Example 2 (a singular perturbation problem)

$$\epsilon x^2 + 2x - 1 = 0$$

Exact solutions are

$$x_\pm = \frac{-2 \pm \sqrt{4+4\epsilon}}{2\epsilon} = \frac{-1 \pm \sqrt{1+\epsilon}}{\epsilon}$$

Again, suppose one only know asymptotic analysis, he would try

$$x = x_0\epsilon^\alpha + o(\epsilon^\alpha)$$

work out the computation and determine x_0, alpha