# Hopf Bifurcation

## 1 Hopf Bifurcation (§6.5)

### 1.1 Motivation: Birth of Oscillations from Steady States

In Lecture 9, we studied bifurcations where steady states exchange stability (e.g., saddle-node, pitchfork). Now we consider a fundamentally different phenomenon: a **steady state losing stability** and giving birth to a **limit cycle** (stable periodic orbit). This is called a **Hopf bifurcation**, named after Eberhard Hopf (1942).

**Remark 1.** *Unlike pitchfork or transcritical bifurcations, Hopf bifurcation involves a change in the dynamical (not just static) nature of the solution: from equilibrium to oscillation.*

### 1.2 The van der Pol oscillator as a prototype

Consider the van der Pol equation:

$$y'' - \lambda(1 - y^2)y' + y = 0, \quad \lambda \in \mathbb{R}. \tag{6.48}$$

This models an oscillator with **nonlinear damping**:

- For small $y$ ($|y| < 1$): damping coefficient $\lambda(1 - y^2) > 0$ if $\lambda > 0 \Rightarrow$ *negative damping* $\Rightarrow$ growth.
- For large $y$ ($|y| > 1$): damping becomes positive $\Rightarrow$ saturation.
- For $\lambda < 0$: damping is negative for large $|y|$, positive for small $|y| \Rightarrow$ different stability picture.

### 1.3 Linearized Stability Analysis of the Trivial Solution

Let $y_s = 0$ be the steady state. Consider a small perturbation:

$$y(t) = \delta y_1(t), \quad \delta \ll 1, \quad y(0) = \alpha_0\delta, \quad y'(0) = \beta_0\delta.$$

Substituting into (6.48) and keeping only $O(\delta)$ terms yields the *linearized equation*:

$$y_1'' - \lambda y_1' + y_1 = 0. \tag{6.49}$$

Assuming $y_1(t) = e^{rt}$ gives the characteristic equation:

$$r^2 - \lambda r + 1 = 0 \quad \Rightarrow \quad r_\pm = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

**Stability Classification**

- $\lambda < 0$: $\mathrm{Re}(r_\pm) < 0 \Rightarrow$ **asymptotically stable** node or focus.
- $\lambda = 0$: $r_\pm = \pm i \Rightarrow$ **center** (neutral stability).
- $\lambda > 0$: $\mathrm{Re}(r_\pm) > 0 \Rightarrow$ **unstable** node or focus.

## 1.4 The Hopf Bifurcation Condition

At $\lambda = 0$, the eigenvalues cross the imaginary axis:

$$r_\pm = \pm i, \quad \text{with} \quad \frac{d}{d\lambda}\text{Re}(r_\pm)\Big|_{\lambda=0} = \frac{1}{2} \neq 0.$$

This satisfies two key conditions for a **standard Hopf bifurcation**:

1. **Non-hyperbolicity**: $\text{Re}(r_\pm) = 0$ at $\lambda = 0$.
2. **Transversality**: The crossing of the imaginary axis is non-degenerate (nonzero speed).

**Remark 2.** *The linear analysis predicts instability for $\lambda > 0$ but cannot predict the long-term fate of solutions. Nonlinear terms determine whether trajectories blow up or settle into a limit cycle.*

## 1.5 Multiple-Scale Analysis Near the Bifurcation Point

To capture the nonlinear saturation, we use the method of multiple scales. Set $\varepsilon = \lambda$ (since $\lambda_b = 0$) and define:

$$t_1 = t \quad \text{(fast time)}, \quad t_2 = \varepsilon t \quad \text{(slow time)}.$$

Expand:

$$y(t) \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \cdots .$$

The $O(1)$ problem gives:

$$\frac{\partial^2 y_0}{\partial t_1^2} + y_0 = 0 \quad \Rightarrow \quad y_0 = A(t_2)\cos(t_1 + \phi(t_2)).$$

At $O(\varepsilon)$, we obtain:

$$\begin{aligned}\frac{\partial^2 y_1}{\partial t_1^2} + y_1 &= (1 - y_0^2)\partial_{t_1} y_0 - 2\partial_{t_1}\partial_{t_2} y_0 \\ &= 2\frac{\partial A}{\partial t_2}\sin(t_1+\phi) + 2A\frac{\partial \phi}{\partial t_2}\cos(t_1+\phi) - A(1 - \tfrac{1}{4}A^2)\sin(t_1+\phi) + \text{(higher harmonics)}.\end{aligned}$$

Removal of secular terms yields the **amplitude equation**:

$$2\frac{dA}{dt_2} = A\left(1 - \frac{1}{4}A^2\right), \quad \frac{d\phi}{dt_2} = 0. \tag{6.56}$$

Solving with initial condition $A(0) = A_0$:

$$A(t_2) = \frac{2}{\sqrt{1 + \left(\frac{4}{A_0^2} - 1\right)e^{-t_2}}}.$$

Thus,

$$y_0 = \frac{2}{\sqrt{1 + ce^{-\varepsilon t}}}\cos(t + \phi) = \frac{2}{\sqrt{1 + ce^{-\lambda t}}}\cos(t + \phi).$$

For $\lambda > 0$ and $t \to \infty$,

$$y(t) \sim 2\cos(t + \phi_0).$$

As $t_2 \to \infty$, $A \to 2$ for any $A_0 > 0$. Thus:

$$y(t) \sim 2\cos(t + \phi_0) \quad \text{as } t \to \infty \text{ for } \lambda > 0.$$

This is a **stable limit cycle** with amplitude 2 and frequency 1.

# 2 Systems of Ordinary Differential Equations (§6.6)

## 2.1 General Framework for Stability Analysis

Consider an $n$-dimensional autonomous system:

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\lambda, \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^n.$$

Let $\mathbf{y}_s$ be a steady state: $\mathbf{f}(\lambda, \mathbf{y}_s) = 0$. Linearize about $\mathbf{y}_s$ by setting $\mathbf{y} = \mathbf{y}_s + \delta \mathbf{v}(t)$ with $\delta \ll 1$:

$$\frac{d\mathbf{v}}{dt} = \mathbf{A}\mathbf{v}, \quad \mathbf{A} = D_{\mathbf{y}}\mathbf{f}(\lambda, \mathbf{y}_s) \in \mathbb{R}^{n\times n}.$$

Assume $\mathbf{v}(t) = \mathbf{x}e^{rt}$ to obtain the eigenvalue problem:

$$\mathbf{A}\mathbf{x} = r\mathbf{x}.$$

## 2.2 Stability Theorem for Linear Systems

**Theorem 2.1** (Linear Stability)**.** *The steady state $\mathbf{y}_s$ is:*

- ***Asymptotically stable*** *if all eigenvalues of $\mathbf{A}$ satisfy $Re(r) < 0$.*
- ***Unstable*** *if any eigenvalue has $Re(r) > 0$.*
- ***Neutrally stable*** *(center) if all eigenvalues have $Re(r) \le 0$ and some have $Re(r) = 0$ but are simple.*

For $2 \times 2$ systems, the characteristic polynomial is $P(s) = s^2 - \text{tr}(\mathbf{A})s + \det(\mathbf{A})$. Stability conditions become:

$$\text{tr}(\mathbf{A}) < 0 \quad \text{and} \quad \det(\mathbf{A}) > 0.$$

## 2.3 Algebraic Stability Criteria: Routh–Hurwitz

For higher-order systems, computing eigenvalues explicitly can be difficult. The **Routh–Hurwitz criterion** provides an algebraic test for stability.

### Hurwitz Matrix Construction

For the characteristic polynomial $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ with $a_n > 0$, form the $n \times n$ Hurwitz matrix:

$$H = \begin{pmatrix} a_{n-1} & a_n & 0 & 0 & \cdots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & \cdots & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}.$$

Let $\Delta_k$ denote the $k$th leading principal minor of $H$.

**Theorem 2.2** (Routh–Hurwitz)**.** *All roots of $P(s)$ have negative real parts if and only if*

$$\Delta_k > 0 \quad \text{for all } k = 1, 2, \ldots, n.$$

### Example: Cubic Polynomial

For $P(s) = s^3 + a_2 s^2 + a_1 s + a_0$, the Hurwitz matrix is:

$$H = \begin{pmatrix} a_2 & 1 & 0 \\ a_0 & a_1 & a_2 \\ 0 & 0 & a_0 \end{pmatrix}.$$

The stability conditions are:

$$a_2 > 0, \quad a_0 > 0, \quad \text{and} \quad a_2 a_1 - a_0 > 0.$$

### 2.4 Example: Hopf Bifurcation in a Two-Dimensional System

Consider the system:

$$y' = v - y[v^2 + y^2 - \lambda(1-\lambda)],$$
$$v' = -y - v[v^2 + y^2 - \lambda(1-\lambda)].$$

The origin $(y_s, v_s) = (0, 0)$ is a steady state for all $\lambda$. The Jacobian at the origin is:

$$\mathbf{A} = \begin{pmatrix} \lambda(1-\lambda) & 1 \\ -1 & \lambda(1-\lambda) \end{pmatrix}.$$

Eigenvalues: $r_\pm = \lambda(1-\lambda) \pm i$.

- For $\lambda < 0$ or $\lambda > 1$: $\mathrm{Re}(r_\pm) < 0 \Rightarrow$ stable focus.
- For $0 < \lambda < 1$: $\mathrm{Re}(r_\pm) > 0 \Rightarrow$ unstable focus.
- At $\lambda = 0$ and $\lambda = 1$: $\mathrm{Re}(r_\pm) = 0 \Rightarrow$ Hopf bifurcation points.

### 2.5 Multiple-Scale Analysis Near $\lambda = 0$

Set $\varepsilon = \lambda$, introduce scales $t_1 = t$, $t_2 = \varepsilon t$, and expand:

$$y \sim \varepsilon^{1/2} y_0(t_1, t_2) + \varepsilon^{3/2} y_1(t_1, t_2) + \cdots.$$

At leading order, we find:

$$y_0 = A(t_2)\sin(t_1 + \theta(t_2)), \quad v_0 = A(t_2)\cos(t_1 + \theta(t_2)).$$

Removing secular terms at $O(\varepsilon^{3/2})$ gives $\theta$ constant and the amplitude equation:

$$\frac{dA}{dt_2} = A(1 - A^2).$$

Thus, as $t \to \infty$, $A \to 1$, yielding the limit cycle:

$$y(t) \sim \sqrt{\lambda}\sin(t + \theta_0), \quad v(t) \sim \sqrt{\lambda}\cos(t + \theta_0) \quad \text{for } 0 < \lambda \ll 1.$$

## Summary of Key Concepts

1. **Hopf bifurcation**: Loss of stability of a steady state through a pair of complex conjugate eigenvalues crossing the imaginary axis, giving birth to a limit cycle.
2. **Transversality condition**: $\frac{d}{d\lambda}\mathrm{Re}(r_\pm) \neq 0$ at the bifurcation point ensures non-degeneracy.
3. **Multiple-scale analysis**: Essential for deriving amplitude equations that describe the nonlinear saturation near the bifurcation.
4. **Linear stability for systems**: Determined by eigenvalues of the Jacobian; asymptotically stable if all $\mathrm{Re}(r) < 0$.
5. **Routh–Hurwitz criterion**: Algebraic test for stability without computing eigenvalues explicitly.
6. **Amplitude equation**: Typically takes the form $\frac{dA}{dt} = \alpha A - \beta A^3$, where $\alpha$ changes sign at the bifurcation point.