Asymptotic analysis

Intro

What is asymptotic analysis?

Asymptotic analysis is the study of how mathematical objects — such as functions, sequences, or solutions to equations — behave in a limiting process, typically when some variable or parameter becomes very large, very small, or approaches a critical value.

At its core, it is about building approximations that capture the essential behavior of a problem in such limiting regimes, often when exact solutions are unavailable or too complicated to use.

What equations are included?

Algebraic, differential, partial differential, integral, difference, etc.

Our core is to treat differential equations.

Key Features

1. Approximation by simpler expressions:

   Replace a complicated function or solution with an asymptotic expansion in terms of a small (or large) parameter.

   Example:

   Taylor series $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}$

   Asymptotic expansions of e to the x power as |x| goes to 0

$$e^x \sim 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \quad as \quad |x| \to 0$$

$$e^x = 1 + x + \frac{x^2}{2} + O\left(|x|^3\right) \quad as \quad |x| \to 0$$

$$e^x = 1 + x + \frac{x^2}{2} + o\left(|x|^2\right) \quad as \quad |x| \to 0$$

   Usually, we will use epsilon for a small parameter/variable.

2. Relative accuracy matters:

   Unlike Taylor series (which are local and centered at a point), asymptotic expansions describe how error compares to the approximation as the parameter tends to the limit.

3. Different from convergence:

   An asymptotic expansion need not converge — but the first few terms can still give very accurate approximations.

4. Intractable problems become approachable:

   Asymptotic analysis is especially valuable when exact solutions are difficult, impossible, or impractical to obtain. Even when a closed-form solution exists (but is too complicated to use), asymptotics provides simplified approximations that capture the essential behavior.

Examples:

Many nonlinear ODEs/PDEs have no closed-form solution, but boundary layer or multiple-scales methods yield usable approximations.

Special functions (e.g., Bessel, Airy, Gamma) have exact definitions, but their asymptotics reveal structure in extreme parameter regimes.

5. Applications:

Physics: wave propagation, fluid mechanics, quantum mechanics.
Engineering: boundary layer theory, oscillations.
Applied math: differential equations with small parameters, special functions, integrals.

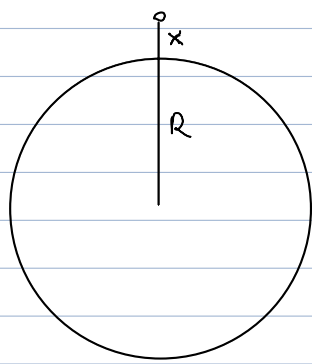
Motivated example:

Projecting an object upward

Key assumptions: spherical planet (for gravity), vacuum (no drag force)

$$F_N = \frac{GMm}{L^2}$$

Newton's second law:



$$\frac{dx^2}{dt^2}m = -\frac{GMm}{(x+R)^2} \qquad mg = \frac{GMm}{R^2},$$

$$\Rightarrow \frac{dx^2}{dt^2} = -\frac{gR^2}{(x+R)^2}$$

Initial conditions:

$$x(0) = x_0 \quad x_t(0) = v_0$$

Nondimensionalization:

$$\tau := t/t_c \quad y(\tau) := x(t)/x_c \quad \text{with} \quad t_c := v_0/g \quad \text{and} \quad x_c := v_0^2/g$$

$$\Rightarrow \frac{dy^2}{d\tau^2} = -\frac{1}{(1+\epsilon y)^2} \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

Observation:

When epsilon is 0, the initial value problem is explicitly solvable, and when epsilon is non-zero, an explicit solution become unavailable due to the nonlinearity.

Guess: Asymptotic expansion of the solution:

$$y(\tau) \sim y_0(\tau) + y_1(\tau)\epsilon + y_2(\tau)\epsilon^2 + \cdots$$

1.3 Order symbols

Taylor's theorem

Theorem 1.1. *Given a function $f(\varepsilon)$, suppose its $(n+1)$st derivative $f^{(n+1)}$ is continuous for $\varepsilon_a < \varepsilon < \varepsilon_b$. In this case, if ε_0 and ε are points in the interval $(\varepsilon_a, \varepsilon_b)$, then*

$$f(\varepsilon) = f(\varepsilon_0) + (\varepsilon - \varepsilon_0) f'(\varepsilon_0) + \cdots + \frac{1}{n!}(\varepsilon - \varepsilon_0)^n f^{(n)}(\varepsilon_0) + R_{n+1}, \quad (1.5)$$

where

$$R_{n+1} = \frac{1}{(n+1)!}(\varepsilon - \varepsilon_0)^{n+1} f^{(n+1)}(\xi) \quad (1.6)$$

and ξ is a point between ε_0 and ε.

Applying the theorem to our former example e^epsilon with epsilon being small, say |epsilon| <= epsilon_0, we obtain that

$$\forall \epsilon \in [-\epsilon_0, \epsilon_0], \quad \exists \xi(\epsilon) \quad \text{lying between 0 and } \xi(\epsilon) \quad \text{s. t.} \quad e^{\epsilon} = 1 + \epsilon + \frac{\epsilon^2}{2} + \frac{e^{\xi(\epsilon)}}{6}\epsilon^3$$

We may view the last term as an error term, and bounding the error by

$$\left| e^{\epsilon} - 1 - \epsilon - \frac{\epsilon^2}{2} \right| = \frac{e^{\xi(\epsilon)}}{6}|\epsilon|^3 \le \frac{e^{\epsilon_0}}{6}|\epsilon|^3 \le \frac{1}{3}|\epsilon|^3, \quad \forall \epsilon \in [-\epsilon_0, \epsilon_0]$$

given that epsilon_0 is a (fixed) number so that $e^{\epsilon_0} \le 2$ for instance $\epsilon_0 = 1/2$

The above motivates the following definition of big O and little o symbols.

Definition 1.1.

1. $f = O(\phi)$ as $\varepsilon \downarrow \varepsilon_0$ means that there are constants k_0 and ε_1 (independent of ε) so that
$$|f(\varepsilon)| \le k_0 |\phi(\varepsilon)| \quad \text{for } \varepsilon_0 < \varepsilon < \varepsilon_1.$$
We say that "f is big Oh of ϕ" as $\varepsilon \downarrow \varepsilon_0$.

2. $f = o(\phi)$ as $\varepsilon \downarrow \varepsilon_0$ means that for every positive δ there is an ε_2 (independent of ε) so that
$$|f(\varepsilon)| \le \delta |\phi(\varepsilon)| \quad \text{for } \varepsilon_0 < \varepsilon < \varepsilon_2.$$
We say that "f is little oh of ϕ" as $\varepsilon \downarrow \varepsilon_0$.

Clear is that

$$e^x = 1 + x + \frac{x^2}{2} + O\left(x^3\right) \quad as \quad x \downarrow 0$$

$$e^x = 1 + x + \frac{x^2}{2} + o\left(x^2\right) \quad as \quad x \downarrow 0$$

Two criteria

Theorem 1.3.

1. If

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = L, \tag{1.7}$$

where $-\infty < L < \infty$, *then* $f = O(\phi)$ *as* $\varepsilon \downarrow \varepsilon_0$.

2. If

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = 0, \tag{1.8}$$

then $f = o(\phi)$ *as* $\varepsilon \downarrow \varepsilon_0$.

These are merely sufficient conditions as we will see next in a few examples.

1. Show the “exponential” example by applying the theorem. Need L’hospital rule.

2. Show that $\epsilon \sin(\frac{1}{\epsilon}) = O(\epsilon) \quad \text{as} \quad \epsilon \downarrow 0$

3. Transcendentally (exponentially) small term. Show that

$$\text{for an arbitrary } \alpha > 0, \quad e^{-1/\epsilon} = o(\epsilon^\alpha) \quad \text{as} \quad \epsilon \downarrow 0$$

We say in this case that f is transcendentally small with respect to the power functions epsilon^alpha

Some properties

(a) $f = O(1)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow f$ is bounded as $\varepsilon \downarrow \varepsilon_0$.

(b) $f = o(1)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow f \to 0$ as $\varepsilon \downarrow \varepsilon_0$.

(c) $f = o(\phi)$ as $\varepsilon \downarrow \varepsilon_0 \Rightarrow f = O(\phi)$ as $\varepsilon \downarrow \varepsilon_0$ (but not vice versa).

1.4 Asymptotic approximations

Consider approximating $f\left(\varepsilon\right) = \varepsilon^2 + \varepsilon^5$

by

$$g_1\left(\varepsilon\right) = \varepsilon^2, \quad g_2(\epsilon) = \frac{2}{3}\epsilon^2$$

The latter approximation is called a “lousy approximation” for its error is of the same order as the function we are using to approximate $f\left(\varepsilon\right)$